

COMPLETE CLASSIFICATION OF THE POSITIVE SOLUTIONS OF $-\Delta u + u^q = 0$

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ABSTRACT. We study the equation $-\Delta u + u^q = 0$, $q > 1$, in a bounded C^2 domain $\Omega \subset \mathbb{R}^N$. A positive solution of the equation is *moderate* if it is dominated by a harmonic function and *σ -moderate* if it is the limit of an increasing sequence of moderate solutions. It is known that in the subcritical case, $1 < q < q_c = (N+1)/(N-1)$, every positive solution is σ -moderate [31]. More recently Dynkin proved, by probabilistic methods, that this remains valid in the supercritical case for $q \leq 2$, [15]. The question remained open for $q > 2$. In this paper we prove that, for all $q \geq q_c$, every positive solution is σ -moderate. We use purely analytic techniques which apply to the full supercritical range. The main tools come from linear and non-linear potential theory. Combined with previous results, this establishes a 1-1 correspondence between positive solutions and their boundary traces in the sense of [35].

1. INTRODUCTION

In this paper we study boundary value problems for the equation

$$(1.1) \quad -\Delta u + |u|^q \text{sign } u = 0, \quad q > 1$$

in a bounded C^2 domain Ω . We say that u is a solution of this equation if $u \in L^q_{\text{loc}}(\Omega)$ and the equation holds in the sense of distributions. Every solution of the equation is in $W^{2,\infty}_{\text{loc}}(\Omega)$. In particular, every solution is in $C^1(\Omega)$.

Let $\mathfrak{M}(\partial\Omega)$ denote the space of finite Borel measures on the boundary. Put

$$\rho(x) := \text{dist}(x, \partial\Omega)$$

and denote by $L^q_\rho(\Omega)$ the Lebesgue space with weight ρ .

For $\nu \in \mathfrak{M}(\partial\Omega)$ a (classical) weak solution of the boundary value problem

$$(1.2) \quad -\Delta u + |u|^q \text{sign } u = 0 \text{ in } \Omega, \quad u = \nu \text{ on } \partial\Omega$$

is a function $u \in L^1(\Omega) \cap L^q_\rho(\Omega)$ such that

$$(1.3) \quad -\int_\Omega u \Delta \phi \, dx + \int_\Omega |u|^q \text{sign } u \phi \, dx = -\int_{\partial\Omega} \partial_{\mathbf{n}} \phi \, d\nu,$$

for every $\phi \in C^2_0(\bar{\Omega})$ where

$$(1.4) \quad C^2_0(\bar{\Omega}) := \{\phi \in C^2(\bar{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}.$$

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The boundary value problem (1.2) with data given by a finite Borel measure is well understood. It is known that, if a solution exists then it is unique. Gmira and Véron [20] proved that, if $1 < q < (N+1)/(N-1)$, the problem possesses a solution for every $\nu \in \mathfrak{M}(\partial\Omega)$; if $q \geq (N+1)/(N-1)$ then the problem has no solution for any measure ν concentrated at a point. The number $q_c := (N+1)/(N-1)$ is the *critical value* for (1.2). The interval $(1, (N+1)/(N-1))$ is the *subcritical* range; the interval $[(N+1)/(N-1), \infty)$ is the *supercritical* range.

In the early 90's the boundary value problem (1.2) became of great interest due to its relation to branching processes and superdiffusions (see Dynkin [11, 12], Le Gall [23]). At first, the study of the problem concentrated on the characterization of the family of finite measures for which (1.2) possesses a solution. This question is closely related to the characterization of removable boundary sets. A compact set $K \subset \partial\Omega$ is *removable* if every positive solution u of (1.1) which has a continuous extension to $\bar{\Omega} \setminus K$ can be extended to a function in $C(\bar{\Omega})$.

In a succession of works by Le Gall [24, 25] (for $q = 2$), Dynkin and Kuznetsov [16, 17] (for $1 < q \leq 2$) and Marcus and Véron [32, 33] (the first for $q \geq 2$, the second providing a new proof for all $q \geq q_c$) the following results were established.

Theorem A. *Let K be a compact subset of $\partial\Omega$. Then*

$$(1.5) \quad K \text{ is removable} \iff C^{2/q, q'}(K) = 0.$$

Here $q' = q/(q-1)$ and $C^{2/q, q'}$ denotes Bessel capacity on $\partial\Omega$.

Theorem B. *Let $\nu \in \mathfrak{M}(\partial\Omega)$. Problem (1.2) possesses a solution if and only if $\nu \prec C^{2/q, q'}$, i.e. ν vanishes on every Borel set $E \subset \partial\Omega$ such that $C^{2/q, q'}(E) = 0$.*

Remark A.1. For solutions in $L^q_\rho(\Omega)$, the removability criterion applies to signed solutions as well.

Remark A.2. For a non-negative solution u of (1.1), the removability criterion can be extended to an arbitrary set $E \subset \partial\Omega$. Suppose that u vanishes on every $C_{2/q, q'}$ -finely open subset of $\partial\Omega \setminus E$. Then

$$C^{2/q, q'}(E) = 0 \implies u = 0.$$

This is a consequence of the capacity estimates of [34].

In view of the estimates of Keller [22] and Osserman [39] equation (1.1) possesses solutions which are not in $L^q_\rho(\Omega)$. In particular the equation possesses solutions which blow up everywhere on the boundary (recall that we assume that Ω is of class C^2). Such solutions, called *large solutions* have been studied for a long time (see e.g. Loewner and Nirenberg [28] who studied the case $q = (N+2)/(N-2)$). It was established that the large solution is unique and its asymptotic behavior at the boundary was described (see Bandle and Marcus [7, 8] and the references therein). The uniqueness of

large solutions was also established for domains of class C^0 and even for $C_{2/q,q'}$ -finely open sets (see Marcus and Véron [29, 37]).

The next question in the study of equation (1.1) was whether it is possible to assign to arbitrary solutions a measure, not necessarily finite, which uniquely determines the solution. (Eventually such a measure was called a *boundary trace*.) In investigating this question, attention was restricted to positive solutions. The Herglotz theorem for positive harmonic functions served as a model. But, in contrast to the linear case, here one must allow unbounded measures.

In [24] Le Gall studied (1.1) with $q = 2$ and Ω a disk in \mathbb{R}^2 . He showed that, in this case, every positive solution possesses a boundary trace which uniquely determines the solution. The boundary trace was described in probabilistic terms and the proof relied mainly on probabilistic techniques.

In [30] Marcus and Véron introduced a notion of boundary trace (later Dynkin called it ‘the rough trace’) which can be described as a (possibly unbounded) Borel measure ν with the following properties. There exists a closed set $F \subset \partial\Omega$ such that

- (i) $\nu(E) = \infty$ for every non-empty Borel subset of F ,
- (ii) ν is a Radon measure on $\partial\Omega \setminus F$.

Let us denote the family of positive measures possessing these properties by $\mathcal{B}_{\text{reg}}(\partial\Omega)$. Given a positive solution u of (1.1), we say that it has (rough) boundary trace $\nu \in \mathcal{B}_{\text{reg}}(\partial\Omega)$ if (with F as above)

- (i') For every open neighborhood Q of F , $u \in L^1(\Omega \setminus A) \cap L^q_\rho(\Omega \setminus A)$ and (1.3) holds for every $\varphi \in C_0^2(\bar{\Omega})$ vanishing in a neighborhood of F .
- (ii') If $\xi \in F$ then, for every open neighborhood A of ξ ,

$$\int_{A \cap \Omega} u^q \rho \, dx = \infty.$$

The following result (announced in [30]) was proved in [31]:

Theorem C. *Every positive solution of (1.1) possesses a boundary trace in $\mathcal{B}_{\text{reg}}(\partial\Omega)$.*

If $1 < q < q_c$ then, for every $\nu \in \mathcal{B}_{\text{reg}}(\partial\Omega)$, (1.1) possesses a unique solution with boundary trace ν .

In the supercritical case it was shown in [32] that, under some additional conditions on ν , – mainly that ν must vanish on subsets of $\partial\Omega \setminus F$ of $C_{2/q,q'}$ -capacity zero, – (1.1) possesses a solution with rough trace ν . These conditions were shown to be necessary and sufficient for existence. However, it soon became apparent that in the supercritical case, the solution is no longer unique. A counterexample to this effect was constructed by Le Gall in 1997. Therefore, in order to deal with the super critical case, a more refined definition of boundary trace was necessary.

Kuznetsov [21] and Dynkin and Kuznetsov [18] provided such a definition, which they called ‘the fine trace’. Their definition was similar to that of the

rough trace, but the singular set F was not required to be closed in the Euclidean topology. Instead it was required to be closed with respect to a finer topology defined in probabilistic terms. With this definition they showed that, if $q \leq 2$ then, for any positive ‘fine trace’ ν , (1.1) possesses a solution the trace of which is equivalent, but not necessarily identical, to ν . The equivalence is defined in terms of polarity. Furthermore they showed that the minimal solution corresponding to a prescribed trace is σ -moderate and it is *the unique solution in this class*. The restriction to $q \leq 2$ is due to the fact that the proof was based on probabilistic techniques which do not apply to $q > 2$.

A σ -moderate solution was defined as the limit of an increasing sequence of positive moderate solutions. We recall that a moderate solution is a solution in $L^1(\Omega) \cap L^q_\rho(\Omega)$, i.e., a solution whose boundary trace is a finite measure.

In around the year 2002, Mselati proved in his Ph.D. thesis (under the supervision of Le Gall) that *for $q = 2$ every positive solution of (1.1) is σ -moderate*. This work appeared in [38]. Mselati used a combination of analytic and probabilistic techniques such as the ‘Brownian snake’ developed by Le Gall [27]. Following this, Dynkin [15] extended Mselati’s result proving:

If $q_c \leq q \leq 2$ then every positive solution of (1.1) is σ -moderate.

Instead of the ‘Brownian snake’ technique, which can be applied only to the case $q = 2$, Dynkin’s proof used new results on Markov processes that are applicable to $q \leq 2$.

At about the same time Marcus and Véron introduced a notion of boundary trace – they called it ‘the precise trace’ – based on the classical notion of $C^{2/q,q'}$ -fine topology (see [1]). A Borel measure ν on $\partial\Omega$ belongs to this family of traces, to be denoted by $\mathcal{F}^{2/q,q'}(\partial\Omega)$, if there exists a $C^{2/q,q'}$ -finely closed set $F \subset \partial\Omega$ such that:

- (i) $\nu(E) = \infty$ for every non-empty Borel subset of F .
- (ii) Every point $x \in \partial\Omega \setminus F$ has a $C_{2/q,q'}$ -finely open neighborhood Q_x such that $\nu(Q_x) < \infty$.
- (iii) If E is a Borel set such that $\nu(E) < \infty$ then ν vanishes on subsets of E of $C^{2/q,q'}$ -capacity zero.

In the subcritical case the $C^{2/q,q'}$ -fine topology is identical to the Euclidean topology and consequently the precise trace coincides with the rough trace.

With this definition they proved [35], by purely analytic methods:

Theorem D. *For every $q \geq q_c$:*

- (a) *Every positive solution of (1.1) possesses a boundary trace $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$.*
- (b) *For every measure $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$, problem (1.2) possesses a σ -moderate solution.*

(c) *The solution is unique in the class of σ -moderate solutions.*

The question whether every positive solution of (1.1) with $q > 2$ is σ -moderate remained open. In the present paper we settle this question proving,

Theorem 1 . *For every $q \geq q_c$, every positive solution of (1.1) is σ -moderate.*

The proof employs only analytic techniques and *applies to all $q \geq q_c$* . Of course the statement is also valid in the subcritical case, in which case it is an immediate consequence of Theorem C.

Combining Theorems C, D with Theorem 1 we obtain:

Corollary 1 . *For every $q > 1$ and every non-negative $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$, problem (1.2) possesses a unique solution. If $1 < q < q_c$, $\mathcal{F}^{2/q,q'}(\partial\Omega) = \mathcal{B}_{\text{reg}}(\partial\Omega)$.*

The method developed in the present paper can be adapted and applied to a general class of problems which includes boundary value problems for equations such as

$$-\Delta u + \rho^\alpha |u|^q \text{sign } u = 0, \quad \alpha > -2$$

and

$$-\Delta u + g(u) = 0,$$

where $g \in C(\mathbb{R})$ is odd, monotone increasing and satisfies the Δ_2 condition and the Keller–Osserman condition. For equations of the latter type, the method can be adapted to boundary value problems in Lipschitz domains as well. These results will be presented in a subsequent paper.

The main ingredients used in the present paper are:

(a) Nonlinear potential theory and fine topologies associated with Bessel capacities (see [1] and [35]).

(b) The theory of boundary value problems for equations of the form

$$L^V u := -\Delta u + Vu = 0 \quad \text{in } \Omega,$$

where $V > 0$ and $\rho^2 V$ is bounded. Here we use mainly the results of Ancona [3] together with classical potential theory results (see e.g. [2]).

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2. PRELIMINARIES: ON THE EQUATION $-\Delta u + Vu = 0$.

For the convenience of the reader we collect here some definitions and results of classical potential theory concerning operators of the form $L^V = -\Delta + V$, that will be used in the sequel. The results apply also to operators of the form $-L_0 + V$ where L_0 is a second order uniformly elliptic operator on

differentiable manifolds with negative curvature. However we shall confine ourselves to the operator L^V in a bounded domain $\Omega \subset \mathbb{R}^N$ which is either a C^2 domain or Lipschitz.

The following conditions on V will be assumed, without further mention, throughout the paper.

$$(2.1) \quad 0 < V \leq c\rho(x)^2, \quad V \in C(\Omega).$$

By [3] if Ω is a bounded Lipschitz domain, the Martin boundary can be identified with the Euclidean boundary $\partial\Omega$ and, for every $\zeta \in \partial\Omega$ there exists a positive L^V harmonic which vanishes on $\partial\Omega \setminus \{\zeta\}$. If normalized this harmonic is unique. We choose a fixed reference point, say $x_0 \in \Omega$ and denote by K_ζ^V this L^V harmonic, normalized by $K_\zeta^V(x_0) = 1$.

We observe that the positivity of V is essential for this result. Indeed the result depends on the weak coercivity of L^V (see definition in [3, Section 2]) which is guaranteed in our case by Hardy's inequality.

As a consequence of the above one obtains the following basic result (see Ancona [3], Theorem 3 and Corollary 13),

Representation Theorem. *For each positive L^V -harmonic function u in Ω there exists a unique positive measure μ on $\partial\Omega$ such that*

$$(2.2) \quad u(x) = \int_{\partial\Omega} K_\zeta^V d\mu(\zeta) \quad \forall x \in \Omega.$$

The function

$$K^V(\cdot, \zeta) = K_\zeta^V(\cdot)$$

is the Martin kernel. In C^2 -domains, with respect to a classical elliptic operator such as $-\Delta$, it can be identified with the Poisson kernel P . More precisely in this case

$$K^0(\cdot, \zeta) = P(\cdot, \zeta)/P(x_0, \zeta),$$

where x_0 is a fixed reference point in Ω . In Lip domains, with respect to $-\Delta$, K^0 is precisely the harmonic measure.

In the sequel we write

$$K_\zeta := K_\zeta^0.$$

The measure μ corresponding to an L^V harmonic u will be called *the L^V boundary trace of u* and we use the notation

$$(2.3) \quad \mathbb{K}_\mu^V := \int_{\partial\Omega} K_\zeta^V d\mu(\zeta), \quad \mathbb{K}_\mu := \int_{\partial\Omega} K_\zeta d\mu(\zeta).$$

Let D be a Lipschitz domain such that $\bar{D} \subset \Omega$ and $h \in L^1(\partial D)$. We denote by $S^V(D, h)$ the solution of the problem

$$(2.4) \quad L^V w := -\Delta w + Vw = 0 \text{ in } D, \quad w = h \text{ on } \partial D.$$

If μ is a finite measure on ∂D , $S^V(D, \mu)$ is defined in the same way. If D is a C^2 domain, a function $w \in L^1(D)$ is a solution of (2.4) (with h replaced

by μ) if

$$(2.5) \quad \int_D (-w\Delta\varphi + Vw\varphi)dx = - \int_{\partial D} \partial_{\mathbf{n}}\varphi d\mu,$$

for every $\varphi \in C_0^2(\bar{D})$.

A family of domains $\{\Omega_n\}$ such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\cup \Omega_n = \Omega$ is called an *exhaustion of Ω* . We say that $\{\Omega_n\}$ is a Lipschitz (resp. C^2) exhaustion if each domain Ω_n is Lipschitz (resp. C^2).

An l.s.c. function $u \in L_{\text{loc}}^1(\Omega)$ is L^V superharmonic if $L^V u \geq 0$ in distribution sense. Such a function is necessarily in $W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$ and consequently it possesses an L^1 trace on ∂D for every C^2 domain $D \Subset \Omega$. Furthermore, for every such domain, $u \geq S^V(D, u)$. If u is positive, the same holds for every Lipschitz domain $D \Subset \Omega$.

If u is an L^V -superharmonic in Ω and D a C^2 domain such that $D \Subset \Omega$ then the function u_D defined by

$$u_D = S^V(D, u) \text{ in } D, \quad u_D = u \text{ in } \Omega \setminus D$$

is called the *D-truncation of u* . This function is an L^V -superharmonic.

Lemma 2.1. *Let u be a non-negative L^V -superharmonic and $\{\Omega_n\}$ a Lipschitz exhaustion of Ω . Then the following limit exists*

$$(2.6) \quad \tilde{u} := \lim S^V(\Omega_n, u)$$

and \tilde{u} is the largest L^V -harmonic dominated by u .

Proof. The sequence $\{S^V(\Omega_n, u)\}$ is non-increasing. Consequently the limit exists and it is an L^V -harmonic. Every L^V harmonic v dominated by u must satisfy $v \leq S^V(\Omega_n, u)$ in Ω_n . Therefore \tilde{u} is the largest such harmonic. \square

Definition 2.2. *A non-negative L^V -superharmonic is called an L^V -potential if its largest L^V -harmonic minorant is zero.*

The following is an immediate consequence of Lemma 2.1.

Lemma 2.3. *A non-negative superharmonic function p is an L^V potential if and only if*

$$S^V(D_\beta, p) \rightarrow 0 \text{ as } \beta \rightarrow 0.$$

Riesz decomposition theorem. *Every non-negative L^V -superharmonic u can be written in a unique way in the form $u = p + h$ where p is an L^V potential and h a non-negative L^V -harmonic.*

Remark. In fact $h = \tilde{u}$ as defined in (2.6).

For further results concerning the L^V -potential see [2, Ch.I, sec. 4].

Definition 2.4. *Let $A \subset \Omega$ and let s be a positive L^V -superharmonic. Then R_s^A (called the reduction of s relative to A) is given by*

$$R_s^A = \text{lower envelope of } \{f : 0 \leq f \text{ superharmonic, } s \leq f \text{ on } A.\}$$

If A is open then R_s^A itself is L^V -superharmonic so that the lower envelope is simply the minimum, [2, p.13].

Definition 2.5. Let $\zeta \in \partial\Omega$. A set A is L^V thin at ζ (in French 'A est ζ -effilé') if $R_{K_\zeta^V}^A \not\equiv K_\zeta^V$.

In view of a theorem of Brelot, if A is open:

$$R_{K_\zeta^V}^A \not\equiv K_\zeta^V \iff R_{K_\zeta^V}^A \text{ is an } L^V\text{-potential.}$$

Furthermore, even if A is not open there exists an open set O such that $A \subset O$ and O is thin at ζ .

Lemma 2.6. Assume that A is thin at $\zeta \in \partial\Omega$ and A open. Let $\{D_n\}$ be a C^2 exhaustion of Ω and put $A_n = \partial\Omega_n \cap A$. Then

$$S^V(\Omega_n, K_\zeta^V \chi_{A_n}) \rightarrow 0.$$

Definition 2.7. Let $\zeta \in \partial\Omega$ and f a real function on Ω . We say that f admits the fine limit ℓ at ζ if there exists a closed set $E \subset \Omega$ such that E is thin at ζ and

$$\lim_{x \rightarrow \zeta, x \in \Omega \setminus E} f(x) = \ell.$$

To indicate this type of convergence we write,

$$\lim_{x \rightarrow \zeta} f(x) = \ell, \text{ } L^V\text{-finely.}$$

Recall that there also exists an open set A such that $E \subset A$ and A is thin at ζ .

Proposition 2.8. Let u be a positive L^V harmonic function, or a solution of (1.1). For $\zeta \in \partial\Omega$,

$$\lim_{x \rightarrow \zeta} u = b \quad L^V\text{-finely} \implies \lim_{x \rightarrow \zeta} u = b \text{ n.t.,}$$

where 'n.t.' means 'non-tangentially'.

Proof. Let $\rho(x) := \text{dist}(x, \partial\Omega)$. By [2, Lemma 6.4], if A is an L^V thin set at ζ and $\beta_n \downarrow 0$ then

$$A \cap \{x \in \Omega : |x - \zeta| < \rho(x), \beta_n/2 < \rho(x) < 3/2\beta_n\} \neq \emptyset$$

for all sufficiently large n . Therefore the assertion follows from Harnack's inequality. \square

For the next two theorems see [2, Prop.1.6 & Thm. 1.8].

Theorem 2.9. If p is an L^V potential then, for every positive L^V harmonic v :

$$\lim_{x \rightarrow \zeta, \text{fine}} p/v = 0 \quad \mu_v\text{-a.e.}$$

where μ_v is the L^V boundary trace of v .

Theorem 2.10. [Fatou-Doob-Naim] *If u, v are positive L^V harmonics then u/v admits a fine limit μ_v a.e. Furthermore*

$$\lim_{x \rightarrow \zeta, \text{fine}} u/v = f = \frac{d\mu_u}{d\mu_v} \quad \mu_v - \text{a.e.}$$

where μ_u and μ_v are the L^V boundary traces of u and v respectively and the term on the right hand side denotes the Radon-Nikodym derivative.

The next lemma – an application of the theorem of Fatou – is due to Ancona [5].

Lemma 2.11. *Assume that v is a positive L^V harmonic function With L^V boundary trace ν . Then*

$$(2.7) \quad \lim_{x \rightarrow \zeta} v > 0 \quad \text{n.t. } \nu\text{-a.e. } \zeta \in \partial\Omega.$$

If $\nu \perp \mathbb{H}_{N-1}$ then

$$(2.8) \quad \lim_{x \rightarrow \zeta} v = \infty \quad \text{n.t. } \nu\text{-a.e.}$$

Proof. The function 1 is an L^V superharmonic. If it is a potential then, by Theorem 2.9

$$\lim_{x \rightarrow \zeta} 1/v = 0 \quad L^V\text{-finely } \nu\text{-a.e.}$$

Therefore, by Proposition 2.8

$$\lim_{x \rightarrow \zeta} v = \infty \quad \text{n.t. } \nu\text{-a.e.}$$

If 1 is not a potential there exists a positive L^V harmonic w and a potential p such that $1 = w + p$. Let $w = K_\gamma^V$ and put $d\gamma/d\nu =: f$. By Theorem 2.10

$$\lim w/v = f \quad L^V\text{-finely } \nu\text{-a.e.}$$

(We do not exclude the possibility that $f = 0$ ν -a.e. but, of course, $f < \infty$ ν -a.e.) Since $p/v \rightarrow 0$ finely ν -a.e., it follows that

$$\lim 1/v = \lim(w + p)/v = f \quad L^V\text{-finely } \nu\text{-a.e.}$$

Applying again Proposition 2.8 we obtain

$$\lim 1/v = f \quad \text{n.t. } \nu\text{-a.e.}$$

which in turn implies (2.7).

If $\nu \perp \mathbb{H}_{N-1}$ then $f = 0$ ν -a.e. and consequently $v \rightarrow \infty$ n.t. ν -a.e. □

3. MODERATE SOLUTIONS OF $L^V u = 0$

We recall some definitions from [15] following the notation of [6].

Definition 3.1. We shall say that a boundary point ζ is L^V regular if $\tilde{K}^V(\cdot, \zeta)$ ($=$ the largest L^V harmonic dominated by the L^V superharmonic $K(\cdot, \zeta)$) is positive. The point ζ is L^V singular if $\tilde{K}^V(\cdot, \zeta) = 0$, i.e. $K(\cdot, \zeta)$ is an L^V potential.

We denote by $Sing(V)$, respectively $Reg(V)$, the set of singular points, respectively regular points, of L^V .

Remark. If ζ is L^V -regular then

$$\tilde{K}^V(\cdot, \zeta) = c(x_0)K^V(\cdot, \zeta), \quad c(x_0) = \tilde{K}^V(x_0, \zeta),$$

where x_0 is a fixed reference point in Ω such that

$$K^V(x_0, \zeta) = 1 \quad \forall \zeta \in \partial\Omega.$$

Notation. The family of finite Borel measures on a set A is denoted by $\mathfrak{M}(A)$. For $A = \partial\Omega$ we shall write simply \mathfrak{M} . If $\mu \in \mathfrak{M}(A)$ we denote by $|\mu|$ the total variation measure and by $\|\mu\|_{\mathfrak{M}(A)}$ the total variation norm.

Definition 3.2. A positive L^V superharmonic u is L^V -moderate if

$$(3.1) \quad \int_{\Omega} u V \rho dx < \infty.$$

An L^V harmonic u is L^V -moderate if $u \in L^1(\Omega) \cap L^1(\Omega; V\rho)$ and

$$(3.2) \quad \int_{\Omega} (-u\Delta\varphi + uV\varphi)dx = - \int_{\partial\Omega} \partial_{\mathbf{n}}\varphi d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$.

A measure $\nu \in \mathfrak{M}$ is L^V -moderate if there exists a moderate L^V harmonic satisfying (3.2).

The space of L^V moderate measures is denoted by \mathfrak{M}^V .

Definition 3.3. Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$. We say that u possesses an m -boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$ if, for every C^2 -exhaustion of Ω , say $\{\Omega_n\}$,

$$u|_{\partial\Omega_n} d\mathbb{H}_{N_1} \rightharpoonup \nu,$$

weakly with respect to $C(\bar{\Omega})$, i.e.,

$$(3.3) \quad \int_{\partial\Omega_n} uh dS \rightarrow \int_{\partial\Omega} h d\nu \quad \forall h \in C(\bar{\Omega}).$$

Remark. If u possesses an m -boundary trace ν then $u \in L^1(\Omega)$ and

$$(3.4) \quad \sup \int_{\partial\Omega_n} |u| dS < \infty.$$

This follows immediately from the definition.

Notation. Let $\rho(x) := \text{dist}(x, \partial\Omega)$. In the case of C^2 domains, there exists $\beta_0 > 0$ such that for $x \in \Omega$, $\rho(x) < \beta_0$, there exists a unique point on $\partial\Omega$, to be denoted by $\sigma(x)$, such that

$$|x - \sigma(x)| = \rho(x).$$

Thus

$$x - \sigma(x) = \rho(x)\nu_{\sigma(x)}$$

where ν_ζ denotes the unit normal at $\zeta \in \partial\Omega$ pointing into the domain. (We also denote $-\nu_\zeta =: \mathbf{n}_\zeta$.) It can be shown that the function $x \mapsto \sigma(x)$ is in $C^2(\bar{\Omega}^0)$ where

$$\Omega^0 := \{x \in \Omega : 0 < \rho(x) < \beta_0\}.$$

The mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^2 homeomorphism of Ω^0 onto

$$\{(\rho, \sigma) \in \mathbb{R}_+ \times \partial\Omega : 0 < \rho < \beta_0\}.$$

Thus (ρ, σ) can be used as an alternative set of coordinates in Ω^0 ; we call them ‘flow coordinates’.

Put,

$$D_\beta = [x \in \Omega, \rho(x) > \beta], \quad \Omega_\beta = \Omega - \bar{D}_\beta, \quad \Sigma_\beta = [x \in \Omega, \rho(x) = \beta].$$

and for $\alpha \in (0, \infty)$ and $\zeta \in \partial\Omega$

$$\mathcal{C}_\zeta^\alpha = \{x \in \Omega^0 : |x - \zeta| > \alpha\rho(x)\}.$$

When $\alpha = 1$, the upper index will be omitted.

In the sequel we assume that Ω is a bounded C^2 domain.

Lemma 3.4. (i) If v is a positive L^V moderate superharmonic then $v \in L^1(\Omega)$ and it possesses an m -boundary trace $\nu \in \mathfrak{M}$. The maximal L^V -harmonic dominated by v , say v' , has the same m -boundary trace.

(ii) If v is a positive L^V superharmonic and possesses an m -boundary trace $\nu \in \mathfrak{M}$ then v' , defined as in (i), is L^V moderate and has m -boundary trace ν .

(iii) If v is an L^V harmonic and v possesses an m -boundary trace ν then v is L^V moderate.

Proof. (i) Let $w \in L^1(\Omega)$ be the (unique) solution of the problem

$$(3.5) \quad -\Delta w = Vv \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Then

$$-\Delta(w + v) \geq 0$$

and consequently $w + v \in L^1(\Omega)$ and it possesses an m -boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$. As $w \in L^1(\Omega)$ and has m -boundary trace zero, it follows that $v \in L^1(\Omega)$ and has m -boundary trace ν .

Given $\varphi \in C_0^2(\bar{\Omega})$, for each $\beta \in (0, \beta_0/2)$ we can construct a function $\varphi_\beta \in C_0^2(\bar{D}_\beta)$ such that

$$\varphi_\beta(x) = \varphi(\rho(x) - \beta, \sigma(x)) \text{ for } \beta \leq \rho(x) < \beta + \beta_0/4$$

and

$$\varphi_\beta \rightarrow \varphi \text{ in } C^2(\Omega), \quad \sup_\beta \|\varphi_\beta\|_{C^2(\bar{D}_\beta)} < \infty.$$

Let

$$v_\beta = S^V(D_\beta, v).$$

Then $0 \leq v_\beta \leq v$ and $v_\beta \downarrow v'$ as $\beta \downarrow 0$. Furthermore

$$\int_{D_\beta} (-v_\beta \Delta \varphi_\beta + v_\beta V \varphi_\beta) dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}} \varphi_\beta v dS.$$

Since $v \in L^1(\Omega; V\rho) \cap L^1(\Omega)$ and v possesses an m-trace ν on $\partial\Omega$ we obtain (by going to the limit as $\beta \rightarrow 0$):

$$(3.6) \quad \int_{\Omega} (-v' \Delta \varphi + v' V \varphi) dx = - \int_{\partial\Omega} \partial_{\mathbf{n}} \varphi d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$.

(ii) If v is a positive L^V harmonic then there exists a Radon measure $\lambda > 0$ in Ω such that $L^V u = \lambda$ in the sense of distributions. Therefore $v \in W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$. Let $v_\beta = S^V(D_\beta, v)$. Then

$$(3.7) \quad \int_{D_\beta} (-v_\beta \Delta \varphi + v_\beta V \varphi) dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}} \varphi v dS$$

for every $\varphi \in C^2(\bar{D}_\beta)$. Choosing φ to be the solution of

$$-\Delta \varphi = 1 \text{ in } D_\beta, \quad \varphi = 0 \text{ on } \partial D_\beta$$

we obtain

$$(3.8) \quad \|v_\beta\|_{L^1(D_\beta)} + \|v_\beta\|_{L^1(D_\beta; V\rho_\beta)} \leq C \int_{\partial D_\beta} v dS$$

with a constant C independent of β . Here ρ_β is the first eigenfunction of $-\Delta$ in D_β normalized by $\rho_\beta(x_0) = 1$. Since, by assumption, v has an m-boundary trace, the right hand side of the inequality is bounded. In addition, ρ_β tends to the first normalized eigenfunction of $-\Delta$ in Ω . Therefore, letting $\beta \rightarrow 0$ we obtain $v_\beta \rightarrow v'$, locally uniformly in Ω , $v' \in L^1(\Omega) \cap L^1(\Omega; V\rho)$ and

$$(3.9) \quad \int_{\Omega} (-v_\beta \Delta \varphi + v_\beta V \varphi) dx = - \int_{\partial\Omega} \partial_{\mathbf{n}} \varphi v dS$$

for every $\varphi \in C^2(\bar{\Omega})$. For the last step we use (3.7) with $\varphi = \varphi_\beta$ as in the first part of the proof.

(iii) The proof is essentially the same as that of part (ii) except that, in the present case, the inequality (3.8) is replaced by

$$(3.10) \quad \|v_\beta\|_{L^1(D_\beta)} + \|v_\beta\|_{L^1(D_\beta; V\rho_\beta)} \leq C \int_{\partial D_\beta} |v| dS.$$

This inequality is proved by a standard argument as in e.g. [32]. \square

Lemma 3.5. (i) If $\nu \in \mathfrak{M}^V$ then the solution of (3.2) is unique. It will be denoted by \mathbb{M}_ν^V .

(ii) The space \mathfrak{M}^V is linear and

$$(3.11) \quad 0 \leq \nu \iff 0 \leq \mathbb{M}_\nu^V$$

(iii) Let $\tau \in \mathfrak{M}$ and $\nu \in \mathfrak{M}^V$.

$$(3.12) \quad |\tau| \leq \nu \implies \tau \in \mathfrak{M}^V,$$

$$(3.13) \quad \nu \in \mathfrak{M}^V \implies |\nu| \in \mathfrak{M}^V.$$

(iv) If $\nu \in \mathfrak{M}^V$ then $|\mathbb{M}_\nu^V| \leq \mathbb{M}_{|\nu|}^V$.

Proof. (i), (ii) and (3.12) are classical. We turn to the proof of (3.13). Put $u = \mathbb{M}_\nu^V$. Then there exist v^+ and v^- in $L^1(\Omega) \cap L^1(\Omega; V\rho)$ such that

$$-\Delta v^\pm + Vu_\pm = 0 \text{ in } \Omega, \quad v^\pm = \nu_\pm \text{ on } \partial\Omega.$$

It follows that $u = v^+ - v^-$ and $|u| \leq v^+ + v^- =: w$. Furthermore,

$$-\Delta w + V|u| = 0.$$

Hence w is a moderate L^V superharmonic with m-boundary trace $|\nu|$. By Lemma 3.4, the largest L^V harmonic dominated by w , say w' , has the same m-boundary trace $|\nu|$. Thus $|\nu|$ is L^V moderate. Since $0 < w'$ and $u < w'$ we obtain $|u| < w'$, i.e., (iv). \square

Denote,

$$(3.14) \quad \mathfrak{M}_0^V := \{\nu \in \mathfrak{M} : \mathbb{K}_{|\nu|}V \in L_\rho^1(\Omega)\}.$$

The following is an immediate consequence of Lemma 3.4:

Lemma 3.6. $\mathfrak{M}_0^V \subset \mathfrak{M}^V$.

Proof. If $\nu \in \mathfrak{M}_0^V$ then $\mathbb{K}_{|\nu|}$ is an L^V moderate superharmonic. \square

Remark. In general it is possible that there exists a positive measure ν such that $\mathbb{K}_\nu^V \in L^1(\Omega; V\rho)$ but $\mathbb{K}_\nu \notin L^1(\Omega; V\rho)$.

Lemma 3.7. If $V \in L_\rho^{q'}(\Omega)$ for some $q' > 1$ then every positive measure in $W^{-2/q,q}$ belongs to \mathfrak{M}_0^V .

Proof. If $\nu \in W^{-2/q,q}$ then $\mathbb{K}_\nu \in L_\rho^q(\Omega)$ (see [40, 1.14.4.] or [36]). Therefore $V\mathbb{K}_\nu \in L_\rho^1(\Omega)$. If, in addition, $\nu \geq 0$ then $\nu \in \mathfrak{M}_0^V$. \square

Remark. There are signed measures $\nu \in W^{-2/q,q}$ such that $|\nu| \notin W^{-2/q,q}$. Therefore in general $W^{-2/q,q}$ may not be contained in \mathfrak{M}_0^V .

Proposition 3.8. Let v be a positive, L^V moderate harmonic with m-boundary trace ν . Let ν' be the L^V boundary trace of v , i.e.,

$$(3.15) \quad \mathbb{M}_\nu^V = \mathbb{K}_{\nu'}^V.$$

Then

$$(3.16) \quad \nu'(E) = 0 \iff \nu(E) = 0.$$

Furthermore,

$$(3.17) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \quad \nu' - a.e.$$

Let $F \subset \partial\Omega$ be compact and denote

$$v_F := \inf(v, \mathbb{K}_{\nu_F}), \quad \nu_F := \nu\chi_F.$$

Then v_F is a moderate supersolution of L^V and the largest L^V harmonic dominated by v_F is given by

$$(3.18) \quad (v_F)' = \mathbb{K}_{\nu'_F}^V.$$

Proof. Since $vV \in L_\rho^1(\Omega)$ it follows (by Fubini) that

$$\int_{\partial\Omega} \left(\int_{\Omega} K^V(\cdot, \zeta) V \rho \right) d\nu'(\zeta) < \infty$$

which implies (3.17).

Let F be a compact subset of $\partial\Omega$. If $\nu'(F) > 0$ then $K_{\nu'_F}^V$ (as usual $\nu'_F := \nu'\chi_F$) is a positive L^V moderate harmonic which vanishes on $\partial\Omega \setminus F$ and is dominated by v . Therefore

$$K_{\nu'_F}^V \leq v_F,$$

and consequently $\nu(F) > 0$.

Next we show that $\nu(F) > 0 \implies \nu'(F) > 0$. Since $\nu(F) > 0$, v_F is a positive L^V superharmonic with m-trace ν_F . If $(v_F)'$ is the largest L^V harmonic dominated by v_F then $(v_F)'$ is L^V moderate with m-trace ν_F . Thus $0 < (v_F)' \leq v_F$.

On the other hand, the largest L^V harmonic dominated by v and vanishing on $\partial\Omega \setminus F$ is $\mathbb{K}_{\nu'_F}^V$. It follows that

$$0 < (v_F)' = \mathbb{K}_{\nu'_F}^V.$$

In particular $\nu'(F) > 0$. □

Proposition 3.9. (i) For every $\zeta \in \partial\Omega$,

$$(3.19) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \iff \zeta \text{ is } L^V\text{-regular.}$$

(ii) If ν is a positive measure in \mathfrak{M}^V then $K^V(\cdot, \zeta)$ is L^V -moderate ν -a.e. and

$$(3.20) \quad \nu(\text{Sing}(V)) = 0.$$

(iii) If $V \in L^{q'}(\omega)$ for some $q' > 1$ then $C_{2/q, q'}$ -a.e. point $\zeta \in \partial\Omega$ is L^V regular. (Here $\frac{1}{q} + \frac{1}{q'} = 1$.)

Proof. (i) Assume that $K^V(\cdot, \zeta)V \in L_\rho^1(\Omega)$. Then, by Lemma 3.4, K_ζ^V is L^V moderate. Its m-boundary trace $\tau_\zeta \in \mathfrak{M}$ is concentrated at ζ . Thus $\tau_\zeta = a(\zeta)\delta_\zeta$ for some $a > 0$. It follows that K_ζ^V is a subsolution of the boundary value problem

$$-\Delta z = 0 \text{ in } \Omega, \quad z = \tau_\zeta \text{ on } \partial\Omega.$$

Therefore

$$K^V(\cdot, \zeta) \leq a(\zeta)K(\cdot, \zeta).$$

This implies that $\tilde{K}_\zeta^V > 0$, i.e., ζ is regular.

Assume that ζ is L^V regular. Then, by definition, $K(\cdot, \zeta)$ is not a potential and has the m-boundary trace δ_ζ . By Lemma 3.4 the largest L^V harmonic dominated by $K(\cdot, \zeta)$, which we denote by $\tilde{K}^V(\cdot, \zeta)$, has the same boundary trace and is L^V moderate. By uniqueness of the positive, normalized L^V harmonic vanishing on $\partial\Omega \setminus \{\zeta\}$,

$$K^V(\cdot, \zeta) = \tilde{K}^V(\cdot, \zeta) / \tilde{K}^V(x_0, \zeta).$$

Thus $K^V(\cdot, \zeta)V \in L_\rho^1(\Omega)$.

(ii) By (3.17) and (3.16),

$$(3.21) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \quad \nu - a.e.$$

By (i), this implies the second assertion.

(iii) In this case, every positive measure $\nu \in W^{-2/q, q}(\partial\Omega)$ is in \mathfrak{M}_0^V which is contained in \mathfrak{M}^V . It follows that the set of singular points of L^V must have $C_{2/q, q'}$ -capacity zero. \square

4. PRELIMINARIES: ON THE EQUATION $-\Delta u + u^q = 0$

In this section we collect some definitions and known results on positive solutions of (1.1), that will be needed for the proof of the main result.

A basic concept in this theory is that of $C_{2/q, q'}$ -fine topology. For the general theory of $C_{m, p}$ capacity and $C_{m, p}$ -fine topology we refer the reader to [1]. For more special results, useful in our study, we refer the reader to the summary in [35, Section 2].

The closure of a set $A \subset \partial\Omega$ in $C_{2/q, q'}$ -fine topology will be denoted by \tilde{A} . We shall say that two sets A, B are $C_{2/q, q'}$ equivalent (or briefly q-equivalent) if $C_{2/q, q'}(A \Delta B) = 0$.

There exists a constant c such that for every set A

$$C_{2/q, q'}(\tilde{A}) \leq cC_{2/q, q'}(A).$$

We recall the definition of regular and singular boundary points of a positive solution u of (1.1). A point $\zeta \in \partial\Omega$ is a q -regular point of u if there exists a $C_{2/q, q'}$ neighborhood of ζ , say O_ζ such that

$$\int_{O_\zeta \cap \Omega} u^q \rho dx < \infty.$$

ζ is q-singular if it is not q-regular. The set of q-regular points is denoted by $\mathcal{R}(u)$ and the set of singular points by $\mathcal{S}(u)$. Evidently $\mathcal{R}(u)$ is $C_{2/q, q'}$ open.

If F is a $C_{2/q, q'}$ -finely closed subset of $\partial\Omega$ then there exists an increasing sequence of compact subsets $\{F_n\}$ such that $C_{2/q, q'}(F \setminus F_n) \rightarrow 0$.

If u is a positive solution of (1.1) we say that it vanishes on a $C_{2/q, q'}$ -finely open set $O = \partial\Omega \setminus F$ if it is the limit of an increasing sequence of positive solutions $\{u_n\}$ such that $u_n \in C(\bar{\Omega} \setminus F_n)$ and $u_n = 0$ on $\partial\Omega \setminus F_n$.

The q -support of the boundary trace of u – denoted by $\text{supp}_{\partial\Omega}^q u$ – is the complement of the largest $C_{2/q,q'}$ -finely open subset of $\partial\Omega$ where u vanishes.

Let $\nu \in \mathfrak{M}$. We say that u is a solution of the problem

$$(4.1) \quad -\Delta u + |u|^q \text{sign } u = 0 \text{ in } D, u = \nu \text{ on } \partial D$$

if $u \in L^1(\Omega) \cap L_\rho^q(\Omega)$ and

$$(4.2) \quad -\int_{\Omega} u \Delta \varphi \, dx + \int_{\Omega} |u|^q \text{sign } u \varphi \, dx = -\int_{\partial} \Omega \partial_{\mathbf{n}} \varphi \, d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$. If a solution exists it is unique; it will be denoted by u_ν . If ν is a measure for which a solution exists, we say that it is q -good. The family of q -good measures is denoted by \mathcal{G}_q . It is known that [9] ν is q -good if and only if it vanishes on sets of $C_{2/q,q'}$ capacity zero. Furthermore, a positive measure $\nu \in \mathfrak{M}$ is q -good if and only if it is the limit of an increasing bounded sequence of measures in $W^{-2/q,q}$. In particular a measure $\nu \in \mathfrak{M}$ such that $|\nu| \in W^{-2/q,q}$ is a q -good measure.

A solution u of (1.1) is *moderate* if $u \in L^1(\Omega) \cap L_\rho^q(\Omega)$. A moderate solution possesses a boundary trace $\nu \in \mathfrak{M}$ such that (4.2) holds.

Denote by \mathcal{U}_q the set of positive solutions of (1.1). A solution $u \in \mathcal{U}_q$ is *σ -moderate* if it is the limit of an increasing sequence of moderate solutions.

A compact set $F \subset \partial\Omega$ is q -removable if a non-negative solution of (1.1) vanishing on $\partial\Omega \setminus F$ must vanish in Ω . An arbitrary set $A \subset \partial\Omega$ is q -removable if every compact subset is q -removable. It is known that A is q -removable if and only if $C_{2/q,q'}(A) = 0$ (see [33] and the references therein).

By [34], if $\{u_n\}$ is a sequence of positive solutions of (1.1) then

$$(4.3) \quad C_{2/q,q'}(\text{supp}_{\partial\Omega}^q u_n) \rightarrow 0 \implies u_n \rightarrow 0 \text{ locally uniformly in } \Omega.$$

If F is a $C_{2/q,q'}$ -finely closed subset of $\partial\Omega$, denote

$$U_F = \sup\{u \in \mathcal{U}_q : \text{supp}_{\partial\Omega}^q u \subset F\}.$$

It is well known that U_F is a solution of (1.1) and it vanishes on $\partial\Omega \setminus F$. We call it *the maximal solution relative to F* .

For an arbitrary Borel set $A \subset \partial\Omega$ denote

$$W_A = \sup\{u_\nu : \nu \in W^{-2/q,q}, \nu(\partial\Omega \setminus A) = 0\}$$

It is proved in [34] that

$$W_A = W_{\bar{A}}$$

and, if F is $C_{2/q,q'}$ -finely closed,

$$U_F = W_F.$$

In particular U_F is *σ -moderate*.

If v is a positive supersolution of (1.1) then the set of solutions dominated by it contains a maximal solution:

$$v^\# := \sup\{u \in \mathcal{U}_q : u \leq v\} \in \mathcal{U}_q.$$

If v is a positive subsolution of (1.1) then the set of solutions dominating it is non-empty and contains a minimal solution:

$$v_{\#} := \inf\{u \in \mathcal{U}_q : u \geq v\} \in \mathcal{U}_q.$$

If $u, v \in \mathcal{U}_q$ then $u + v$ is a supersolution, $(u - v)_+$ is a subsolution and we denote

$$u \oplus v = [u + v]^{\#}, \quad u \ominus v = [(u - v)_+]^{\#}.$$

If $u \in \mathcal{U}_q$ and F is a $C_{2/q, q'}$ -finely closed subset of $\partial\Omega$ we denote:

$$[u]_F = \inf(u, U_F)^{\#}.$$

If D is a C^2 subdomain of Ω and $h \in L^1(\partial D)$ we denote by $S_q(D, h)$ the solution of the problem

$$-\Delta u + |u|^q \text{sign } u = 0 \text{ in } D, u = h \text{ on } \partial D.$$

Let $\{\Omega_n\}$ be a C^2 exhaustion of Ω . Then, if v is a positive supersolution,

$$S_q(\Omega_n, v) \downarrow v^{\#}$$

and if v is a positive subsolution

$$S_q(\Omega_n, v) \uparrow v_{\#}.$$

The following definitions were introduced in [35]. A positive Borel measure τ on $\partial\Omega$ (not necessarily bounded) is called a *perfect measure* if it satisfies the following conditions:

(a) τ is outer regular relative to $C_{2/q, q'}$ -fine topology, i.e., for every Borel set E ,

$$\tau(E) = \inf\{\tau(Q) : Q \text{ is } C_{2/q, q'}\text{-finely open, } E \subset Q\}.$$

(b) If Q is a $C_{2/q, q'}$ -finely open set and A a Borel set such that $C_{2/q, q'}(A) = 0$ then $\tau(Q) = \tau(Q \setminus A)$.

The space of perfect measures is denoted by \mathcal{M}_q .

We observe that (b) implies:

(b') If Q is a $C_{2/q, q'}$ -finely open set, A a Borel subset such that $C_{2/q, q'}(A) = 0$ and $\tau(Q \setminus A) < \infty$ then $\tau(A) = 0$ and $\tau\chi_Q$ is a q -good measure.

For $\tau \in \mathcal{M}_q$ put

$$Q_{\tau} = \bigcup\{Q : Q \text{ is } C_{2/q, q'}\text{-finely open, } \tau(\tilde{Q}) < \infty\}.$$

If $u \in \mathcal{U}_q$ we say that u has boundary trace $\tau \in \mathcal{M}_q$ if:

(i) $\mathcal{R}(u) = Q_{\tau}$ and

(ii) for every $\xi \in Q_{\tau}$ there exists a $C_{2/q, q'}$ -finely open neighborhood Q such that $[u]_{\tilde{Q}}$ is a moderate solution with boundary trace $\tau\chi_{\tilde{Q}}$.

The boundary trace of u in this sense is called *the precise trace* and is denoted by $\text{tr } u$.

By [35, Theorem 5.11], for every $u \in \mathcal{U}_q$, there exists a sequence $\{Q_n\}$ of $C_{2/q,q'}$ -finely open subsets of $\mathcal{R}(u)$ such that

$$\tilde{Q}_n \subset Q_{n+1}, \quad [u]_{\tilde{Q}_n} \text{ is moderate } \forall n, \quad C_{2/q,q'}(\mathcal{R}(u) \setminus \cup_n Q_n) = 0.$$

Such a sequence is called a *regular decomposition* of $\mathcal{R}(u)$. We denote:

$$(4.4) \quad \begin{aligned} \mathcal{R}_0(u) &= \bigcup_n Q_n, \quad \nu_n = \text{tr}[u]_{\tilde{Q}_n}, \\ u_{\mathcal{R}} &= \lim [u]_{\tilde{Q}_n}, \quad \nu_{\mathcal{R}} = \lim \nu_n. \end{aligned}$$

$u_{\mathcal{R}}$ and $\nu_{\mathcal{R}}$ do not depend on the specific sequence $\{Q_n\}$. In fact (by the theorem cited above)

$$(4.5) \quad [u]_F = [u_{\mathcal{R}}]_F \quad \forall F \text{ } C_{2/q,q'}\text{-finely closed, } F \subset \mathcal{R}(u),$$

and $u \ominus u_{\mathcal{R}}$ vanishes on $\mathcal{R}(u)$.

The following result is proved in [35] (see Theorem 5.16 and the remark following it):

Theorem 4.1. *Every positive solution u of (1.1) possesses a boundary trace $\nu \in \mathcal{B}_q$. Conversely, for every $\nu \in \mathcal{B}_q$ there exists a solution of (1.1) with boundary trace ν . Furthermore there exists a unique σ -moderate solution u of (1.1) with $\text{tr } u = \nu$, namely,*

$$u = u_{\mathcal{R}} \oplus U_{\mathcal{S}(u)}.$$

where $u_{\mathcal{R}}$ is the σ -moderate solution defined in (4.4).

In addition, by [35, Theorem 5.11] we obtain:

Theorem 4.2. *If $u \in \mathcal{U}_q$ then*

$$(4.6) \quad \max(u_{\mathcal{R}}, [u]_{\mathcal{S}}) \leq u \leq u_{\mathcal{R}} + [u]_{\mathcal{S}}.$$

Proof. $v := u \ominus u_{\mathcal{R}}$ vanishes on $\mathcal{R}(u)$, i.e., $\text{supp}_{\partial\Omega}^q v \subset \mathcal{S}(u)$. Thus v is a solution dominated by u and supported in $\mathcal{S}(u)$, which implies that $v \leq [u]_{\mathcal{S}}$. Since $u \leq u_{\mathcal{R}} \leq v$ this implies the inequality on the right hand side of (4.6). The inequality on the left hand side is obvious. \square

We finish this section with the following lemma which is used in the proof of the main result.

Lemma 4.3. *Let $u \in \mathcal{U}_q$ and let A, B be two disjoint $C_{2/q,q'}$ -finely closed subsets of $\partial\Omega$. If $u \text{supp}_{\partial\Omega}^q u \subset A \cup B$ and $[u]_A, [u]_B$ are σ -moderate then u is σ -moderate. Furthermore*

$$(4.7) \quad u = [u]_A \oplus [u]_B = [\max(u_A, u_B)]_{\#}.$$

Proof. Let τ and τ' be q -good positive measures such that $q\text{-supp } \tau \cap q\text{-supp } \tau' = \emptyset$. Then

$$[\max(u_{\tau}, u_{\tau'})]_{\#} = u_{\tau} \oplus u_{\tau'} = u_{\tau+\tau'}.$$

Let $\{\tau_n\}$ and $\{\tau'_n\}$ be increasing sequences of q -good measures such that

$$u_{\tau_n} \uparrow [u]_A, \quad u_{\tau'_n} \uparrow [u]_B.$$

By [35, Theorem 4.4]

$$(4.8) \quad \max([u]_A, [u]_B) \leq u \leq [u]_A + [u]_B.$$

Therefore

$$\max(u_{\tau_n}, u_{\tau'_n}) \leq u \implies u_{\tau_n + \tau'_n} \leq u.$$

On the other hand

$$u - u_{\tau_n + \tau'_n} \leq ([u]_A - u_{\tau_n}) + ([u]_B - u_{\tau'_n}) \downarrow 0.$$

Thus

$$(4.9) \quad \lim u_{\tau_n + \tau'_n} = u$$

so that u is σ -moderate.

Assertion (4.7) is equivalent to the statements: (a) u is the largest solution dominated by $[u]_A + [u]_B$ and (b) u is the smallest solution dominating $\max(u_A, u_B)$. Since the maximal solution U_F of a $C_{2/q, q'}$ -finely closed set $F \subset \partial\Omega$ is σ -moderate:

$$[u]_F = \sup\{v \in \mathcal{U}_q : v \leq u, v \text{ moderate}, \text{supp}_{\partial\Omega}^q v \subset F\}.$$

Suppose that $w \in \mathcal{U}_q$ and

$$u \leq w \leq [u]_A + [u]_B.$$

Then,

$$[w]_A \leq [u]_A, \quad [w]_B \leq [u]_B \implies w \leq [u]_A.$$

Therefore, as $u \leq w$ we obtain,

$$[w]_A = [u]_A, \quad [w]_B = [u]_B.$$

Since u is σ -moderate, these equalities and (4.9) imply that $u = w$. This proves (a); statement (b) is proved in a similar way. \square

5. CHARACTERIZATION OF POSITIVE SOLUTIONS OF $-\Delta u + u^q = 0$.

In this section we present the main result of the paper:

Theorem 5.1. *Every positive solution of (1.1) is σ -moderate.*

The proof is based on several lemmas.

The following notation is used throughout the section: u is a positive solution of (1.1),

$$V := u^{q-1}, \quad L^V = -\Delta v + Vv = 0.$$

Thus V satisfies (2.1) and $L^V u = 0$. Therefore there exists a positive measure $\mu \in \mathfrak{M}$ such that

$$u = \mathbb{K}_\mu^V.$$

For any Borel set $E \subset \partial\Omega$ put

$$\mu_E = \mu \chi_E \quad \text{and} \quad (u)_E = \mathbb{K}_{\mu_E}^V.$$

Lemma 5.2. *Let D be a C^2 domain such that $D \Subset \Omega$ and let $h \in L^1(\partial D)$, $0 \leq h \leq u$. Then*

$$(5.1) \quad S^V(D, h) \leq S_q(D, h).$$

Proof. Put $w := S_q(D, h)$ and $v := S^V(D, h)$. Then $w \leq u$ and consequently (recall that $V = u^{q-1}$)

$$0 = -\Delta w + w^q \leq -\Delta w + Vw.$$

Thus w is an L^V superharmonic in D such that $u = h$ on ∂D . On the other hand v is an L^V harmonic in D satisfying the same boundary condition. This implies (5.1). \square

Lemma 5.3. *If F is a compact subset of $\partial\Omega$ then*

$$(5.2) \quad (u)_F \leq [u]_F.$$

Proof. Let A be a Borel subset of $\partial\Omega$. Put

$$A_\beta = \{x \in \Omega : \rho(x) = \beta, \sigma(x) \in A\}$$

and

$$v_\beta^A = S^V(D_\beta, u\chi_{A_\beta}), \quad w_\beta^A = S_q(D_\beta, u\chi_{A_\beta}).$$

By Lemma 5.2, $v_\beta^A \leq w_\beta^A \leq u$. For any sequence $\{\beta_n\}$ decreasing to zero one can extract a subsequence $\{\beta_{n'}\}$ such that $\{w_{\beta_{n'}}^A\}$ and $\{v_{\beta_{n'}}^A\}$ converge locally uniformly; we denote the limits by w^A and v^A respectively. (The limits may depend on the sequence.) Then w^A is a solution of (1.1) while v^A is an L^V harmonic, and

$$(5.3) \quad v^A \leq w^A \leq [u]_{\tilde{Q}} \quad \forall Q \text{ open}, A \subset Q.$$

The second inequality follows from the fact that $w^A \leq u$ and w^A vanishes on $\partial\Omega \setminus \tilde{Q}$.

We apply the same procedure to the set $B = \partial\Omega \setminus A$ extracting a further subsequence of $\{\beta_{n'}\}$ in order to obtain the limits v^B and w^B . Thus

$$v^B \leq w^B \leq [u]_{\tilde{Q}'} \quad \forall Q' \text{ open}, B \subset Q'.$$

Note that

$$v^A + v^B = u, \quad v^A \leq \mathbb{K}_{\mu_{\tilde{Q}}}^V, \quad v^B \leq \mathbb{K}_{\mu_{\tilde{Q}'}}^V.$$

Therefore

$$(5.4) \quad v^A = u - v^B \geq \mathbb{K}_{\mu_{\partial\Omega \setminus \tilde{Q}'}}^V.$$

Now, given F compact, let A be a closed set and O an open set such that $F \subset O \subset A$ and let $B = \partial\Omega \setminus A$. Note that $\bar{B} \cap F = \emptyset$. By (5.4) with $Q' = B$

$$v^A \geq \mathbb{K}_{\mu_O}^V.$$

By (5.3)

$$v^A \leq w^A \leq [u]_{\tilde{Q}} \quad \forall Q \text{ open}, A \subset Q$$

and consequently

$$(5.5) \quad (u)_F \leq \mathbb{K}_{\mu_{\chi_O}}^V \leq [u]_{\tilde{Q}}.$$

If Q shrinks to F then $[u]_{\tilde{Q}} \downarrow [u]_F$ (see [35, Theorem 4.4]). Therefore (5.5) implies (5.2). \square

Lemma 5.4. *If $E \subset \partial\Omega$ is a Borel set and $C_{2/q,q'}(E) = 0$ then $\mu(E) = 0$.*

Proof. If F is a compact subset of E , $C_{2/q,q'}(F) = 0$ and therefore the removability theorem [33] implies that $[u]_F = 0$. Therefore, by Lemma 5.3, $(u)_F = 0$. Consequently $\mu(F) = 0$. As this holds for every compact subset of E we conclude that $\mu(E) = 0$. \square

Lemma 5.5. *Let $\nu \in W^{-2/q,q}(\partial\Omega)$ be a positive measure and let u_ν be the solution of (1.1) with trace ν . Suppose that there exists no positive solution of (1.1) dominated by the supersolution $v = \inf(u, \mathbb{K}_\nu)$. Then $\mu \perp \nu$.*

Proof. First we verify that: if $V' := v^{q-1}$ then v is an $L^{V'}$ superharmonic and furthermore it is an $L^{V'}$ potential.

Indeed v is a supersolution of (1.1) and so

$$0 \leq -\Delta v + v^q = -\Delta v + V'v.$$

Suppose that there exists a positive $L^{V'}$ harmonic w such that $w \leq v$. Then w is a subsolution of (1.1):

$$-\Delta w + w^q \leq -\Delta w + V'w = 0.$$

This implies that there exists a positive solution of (1.1) dominated by v , contrary to assumption. Thus v is an $L^{V'}$ -potential.

Note that

$$\int_{\partial\Omega} \mathbb{K}_\nu V' \rho \, dx \leq \int_{\partial\Omega} (\mathbb{K}_\nu)^q \rho \, dx < \infty.$$

Therefore \mathbb{K}_ν is an $L^{V'}$ moderate superharmonic. Consequently there exists a positive $L^{V'}$ moderate harmonic, say w with m-boundary trace ν such that

$$\mathbb{K}_\nu = w + p$$

where p is an $L^{V'}$ -potential. w can be represented in the form

$$w = \mathbb{K}_{\nu'}^{V'}$$

where ν' is a positive finite measure on $\partial\Omega$ and ν, ν' are mutually a.c.

By the relative Fatou theorem, since v, p are $L^{V'}$ potentials and w is an $L^{V'}$ harmonic,

$$(5.6) \quad v/w \rightarrow 0, \quad \mathbb{K}_\nu/w \rightarrow 1 \quad L^{V'} - \text{finely } \nu' \text{-a.e.}$$

Since $v = \inf(u, \mathbb{K}_\nu)$, (5.6) implies that

$$(5.7) \quad u/w \rightarrow 0 \quad L^{V'} - \text{finely } \nu' \text{-a.e.}$$

Further, by reqfine-nu and (5.7)

$$(5.8) \quad u/\mathbb{K}_\nu \rightarrow 0 \text{ } L^{V'}\text{-finely } \nu'\text{-a.e.}$$

Since ν, ν' are mutually a.c., ' ν -a.e.' is equivalent to ' ν' -a.e.'. Therefore, in view of Proposition 2.8, (5.8) implies

$$(5.9) \quad u/\mathbb{K}_\nu \rightarrow 0 \text{ n.t. } \nu\text{-a.e.}$$

However, \mathbb{K}_ν is also an L^V superharmonic. Therefore \mathbb{K}_ν can be represented in the form

$$\mathbb{K}_\nu = w^* + p^*,$$

where w^* is an L^V -harmonic and p^* an L^V -potential. Let $\tau \in \mathfrak{M}$ be the L^V trace of w^* , i.e., $w^* = \mathbb{K}_\tau^V$. Then, by the relative Fatou theorem,

$$\mathbb{K}_\nu/u \rightarrow \frac{d\tau}{d\mu} =: h \text{ } L^V\text{-finely, } \mu\text{-a.e.}$$

and therefore, by Proposition 2.8,

$$(5.10) \quad \mathbb{K}_\nu/u \rightarrow h \text{ n.t. } \mu\text{-a.e.}$$

Since $0 \leq h < \infty$ μ -a.e., (5.9) and (5.10) imply that $\nu \perp \mu$. \square

Lemma 5.6. *Suppose that for every positive measure $\nu \in W^{-2/q,q}(\partial\Omega)$, there exists no positive solution of (1.1) dominated by $v = \inf(u, \mathbb{K}_\nu)$. Then $u = 0$.*

Proof. By Lemma 5.5,

$$\mu \perp \nu \quad \forall \nu \in W^{-2/q,q}(\partial\Omega), \nu \geq 0.$$

Suppose that $\mu \neq 0$. By Lemma 5.4, μ vanishes on sets of $C_{2/q,q'}$ zero. Therefore (by Feyel and de la Pradelle [19] or Dal Maso [10]) μ is the limit of an increasing sequence $(\mu_k) \subset W_+^{-2/q,q}$. For every k there exists a Borel set $A_k \subset \partial\Omega$ such that,

$$\mu(A_k) = 0, \quad \mu_k(\partial\Omega \setminus A_k) = 0.$$

Therefore, if $A = \cup A_k$ and $A' = \partial\Omega \setminus A$ then

$$\mu(A) = 0, \quad \mu_k(A') = 0 \quad \forall k.$$

Since $\mu_k \leq \mu$ we have $\mu_k(A) = 0$ and therefore $\mu_k = 0$. Contradiction! \square

PROOF OF THEOREM 5.1 Let $\{Q_n\}$ be a regular decomposition of $\mathcal{R}(u)$ and put

$$v_n := [u]_{\tilde{Q}_n}.$$

Using the notation introduced in (4.4), v_n is moderate with boundary trace ν_n and

$$v_n \uparrow u_{\mathcal{R}}.$$

Thus the solution $u_{\mathcal{R}}$ is σ -moderate and

$$u \ominus u_{\mathcal{R}} \leq [u]_{\mathcal{S}(u)} =: u_{\mathcal{S}}.$$

Assertion 1 $u_{\mathcal{S}}$ is σ -moderate.

Before proving the assertion we verify that it implies that u is σ -moderate. Put

$$u_n := v_n \oplus u_{\mathcal{S}}.$$

By Lemma 4.3, as $\tilde{Q}_n \cap \mathcal{S}(u) = \emptyset$, it follows that u_n is σ -moderate. As $\{u_n\}$ is increasing it follows that $\bar{u} = \lim u_n$ is a σ -moderate solution of (1.1). In addition

$$[\max(v_n, u_{\mathcal{S}})]_{\#} = u_n = v_n \oplus u_{\mathcal{S}} \implies \max(u_{\mathcal{R}}, u_{\mathcal{S}}) \leq \bar{u} \leq u_{\mathcal{R}} + u_{\mathcal{S}}.$$

This further implies that $\mathcal{S}(u) = \mathcal{S}(\bar{u})$ and that $\text{tr } \bar{u} = \text{tr } u$. By uniqueness of the σ -moderate solution we conclude that $u = \bar{u}$.

We turn to the proof of *Assertion 1*. To simplify notation, we put $u = u_{\mathcal{S}}$ and denote $F := \text{supp}_{\partial\Omega}^q u$. (Incidentally, $F \subset \mathcal{S}(u)$ but it is possible that there is no equality. In fact F consists precisely of the $C_{2/q,q'}$ -thick points of $\mathcal{S}(u)$. The set $\mathcal{S}(u) \setminus F$ is contained in the singular set of $u_{\mathcal{R}}$.)

For $\nu \in W^{-2/q,q}$ we denote by u_{ν} the solution of (1.1) with boundary trace ν . Put

$$(5.11) \quad u^* = \sup\{u_{\nu} : \nu \in W^{-2/q,q}, 0 < u_{\nu} \leq u\}.$$

By Lemma 5.6 the family over which the supremum is taken is not empty. Therefore u^* is a positive solution of (1.1) and it is well-known that it is σ -moderate. By its definition, $u^* \leq u$.

Let $F^* = \text{supp}_{\partial\Omega}^q u^*$. Then F^* is $C_{2/q,q'}$ -finely closed and $F^* \subset F$. Suppose that $C_{2/q,q'}(F \setminus F^*) > 0$. Then there exists a compact set $E \subset F \setminus F^*$ such that $C_{2/q,q'}(E) > 0$ and $\partial\Omega \setminus F^* =: Q^*$ is a $C_{2/q,q'}$ -finely open set containing E . Furthermore there exists a $C_{2/q,q'}$ -finely open set Q' such that $E \subset Q' \subset \tilde{Q}' \subset Q^*$ ([35, Lemma 2.4]). Since $Q' \subset \text{supp}_{\partial\Omega}^q u$, $[u]_{\tilde{Q}'} > 0$ and therefore, by Lemma 5.6, there exists a positive measure $\tau \in W^{-2/q,q}$ supported in \tilde{Q}' such that $u_{\tau} \leq u$. As the q -supp τ is a $C_{2/q,q'}$ -finely closed set disjoint from F^* it follows that $u^* \not\leq u_{\tau}$. On the other hand, since $\tau \in W^{-2/q,q}$ and $u_{\tau} \leq u$, it follows that $u_{\tau} \leq u^*$. This contradiction shows that

$$(5.12) \quad C_{2/q,q'}(F \setminus F^*) = 0.$$

Further u^* is σ -moderate and therefore there exists a $C_{2/q,q'}$ -finely closed set $F_0^* \subset F^*$ such that $\mathcal{S}(u^*) = F_0^*$ and $\mathcal{R}(u^*) = \partial\Omega \setminus F_0^*$. Suppose that $C_{2/q,q'}(F \setminus F_0^*) > 0$ and put $Q_0 := \partial\Omega \setminus F_0^*$. Let $E \subset F \setminus F_0^*$ be a compact set such that $C_{2/q,q'}(E) > 0$ and let Q' be a $C_{2/q,q'}$ -finely open set such that $E \subset Q' \subset \tilde{Q}' \subset Q_0$. Then $\tilde{Q}' \subset \mathcal{R}(u^*)$ and consequently $[u^*]_{\tilde{Q}'}$ is a moderate solution of (1.1), i.e.

$$[u^*]_{\tilde{Q}'} \in L_{\rho}^q(\Omega).$$

On the other hand Q' is a $C_{2/q,q'}$ -finely open neighborhood of E which is a non-empty subset of $F = \text{supp}_{\partial\Omega}^q u$; therefore $[u]_{\tilde{Q}'}$ is a purely singular

solution of (1.1), i.e.,

$$\int_{\Omega} ([u]_{\tilde{Q}'}^q) \rho \, dx = \infty, \quad \mathcal{S}([u]_{\tilde{Q}'}) = \text{supp}_{\partial\Omega}^q [u]_{\tilde{Q}'}.$$

It follows that $v := [[u]_{\tilde{Q}'} - [u^*]_{\tilde{Q}'}]_{\#}$ is a purely singular solution of (1.1).

Let v^* be defined as in (5.11) with u replace by v . Then v^* is a singular, σ -moderate solution of (1.1). Since $v^* \leq u$ and it is σ -moderate it follows that $v^* \leq u^*$. On the other hand, since v^* is singular and $\text{supp}_{\partial\Omega}^q v^* \subset \tilde{Q}' \subset \mathcal{R}(u^*)$ it follows that $u^* \not\leq v^*$, i.e. $(v^* - u^*)_+$ is not identically zero. Since both u^* and v^* are σ -moderate, it follows that there exists $\tau \in W^{-2/q,q}$ such that $u_{\tau} \leq v^*$ but $(u_{\tau} - u^*)_+$ is not identically zero. Therefore $u^* \not\leq \max(u^*, u_{\tau})$. The function $\max(u^*, u_{\tau})$ is a subsolution of (1.1) and the smallest solution above it, which we denote by Z is *strictly larger* than u^* . However $u_{\tau} \leq v^* \leq u^*$ and consequently $Z = u^*$. This contradiction proves that

$$(5.13) \quad C_{2/q,q'}(F \setminus F_0^*) = 0$$

In conclusion, u^* is σ -moderate, $\text{supp}_{\partial\Omega}^q u^* \subset F$ and $F_0^* = \mathcal{S}(u^*)$ is $C_{2/q,q'}$ -equivalent to F . Therefore, by Theorem 4.1, $u^* = U_F$, the maximal solution supported in F . Since, by definition $u^* \leq u$, it follows $u^* = u$. \square

REFERENCES

- [1] Adams D. R. and Hedberg L. I., Function spaces and potential theory, Grundlehren Math. Wissen. **314**, Springer (1996).
- [2] Ancona A., *Theorie du potentiel sur les graphes et les varietes*, in Springer Lecture Notes No. 1427 (ed. P.L. Hennequin) p.1-112 (1988).
- [3] Ancona A., *Negatively curved manifolds, elliptic operators and the Martin boundary*, Annals of Mathematics, Second Series, **125**, 495-536 (1987).
- [4] Ancona A., Personal communication, 2005.
- [5] Ancona A., Personal communication, 2010.
- [6] Ancona A., *A necessary condition for the fine regularity of a boundary point with respect to a Schrödinger equation*, Appendix in 'Boundary value problems with measures', by L.Véron and C.Yarur (preprint).
- [7] Bandle C. and Marcus M. *Sur les solutions maximales de problèmes elliptiques non linéaires*, C. R. Acad. Sci. Paris Ser. I **311** (1990) 9193.
- [8] Bandle C. and Marcus M. *Asymptotic behaviour of solutions and their derivatives for semilinear elliptic problems with blow up on the boundary*, Ann. Inst. Poincaré **12** (1995) 155171.
- [9] Baras P., Pierre M. *Singularités éliminables pour des équations semilinéaires*, Ann. Inst. Fourier **34** (1984), 185-206.
- [10] Dal Maso G. *On the integral representation of certain local functionals*, Ricerche Mat. **32 no. 1** (1983) 85-113.
- [11] Dynkin E. B., *A probabilistic approach to one class of nonlinear differential equations*, Probab. Th, Rel. Fields **89** (1991), 89-115.
- [12] Dynkin E. B., *Superprocesses and partial differential equations*, Ann. Probab **21** (1993), 1185-1262.
- [13] Dynkin E. B., *Stochastic boundary values and boundary singularities for solutions of the equation $Lu = u^{\alpha}$* , J. Funct. Anal. **153** (1998), 147-186.
- [14] Dynkin E. B. *Diffusions, Superdiffusions and Partial Differential Equations*, American Math. Soc., Providence, Rhode Island, Colloquium Publications **50**, 2002.

- [15] Dynkin E. B. *Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations*, American Math. Soc., Providence, Rhode Island, Colloquium Publications **34**, 2004.
- [16] Dynkin E. B. and Kuznetsov S. E. *Superdiffusions and removable singularities for quasilinear partial differential equations*, Comm. Pure Appl. Math. **49**, 125-176 (1996).
- [17] Dynkin E. B. and Kuznetsov S. E. *Trace on the boundary for solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. **350** (1998) 4499-4519.
- [18] Dynkin E. B. and Kuznetsov S. E. *Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations*, Comm. Pure Appl. Math. **51**, 897-936 (1998).
- [19] Feyel D. and de la Pradelle A. *Topologies fines et compactifications associes certains espaces de Dirichlet*, Ann. Inst. Fourier (Grenoble) **27**, 121146 (1977).
- [20] Gmira A. and Veron L., *Boundary singularities of solutions of nonlinear elliptic equations*, Duke J. Math. **64**, 271-324 (1991).
- [21] Kuznetsov S.E. *σ -moderate solutions of $Lu = u^\alpha$ and fine trace on the boundary*, C.R.Acad. Sc. Serie I **326**, 1189-1194 (1998).
- [22] Keller J. B. , *On solutions of $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10**, 503-510 (1957).
- [23] Le Gall J. F., *A class of path-valued Markov processes and its connections with partial differential equations.*, Probab. Th. Rel. Fields **102** (1993), 2546.
- [24] Le Gall J. F., *Solutions positives de $\Delta u = u^2$ dans le disque unité*, C.R. Acad. Sci. Paris, Sér. I **317** (1993), 873878.
- [25] Le Gall J. F., *The Brownian snake and solutions of $\Delta u = u^2$ in a domain*, Probab. Th. Rel. Fields **102**, 393-432 (1995).
- [26] Le Gall J. F., *A probabilistic Poisson representation for positive solutions of $\Delta u = u^2$ in a domain*, Comm. Pure Appl. Math. **50** (1997), 69103.
- [27] Le Gall J. F., *Spatial branching processes, random snakes and partial differential equations*, Birkhäuser, Basel/Boston/Berlin, 1999.
- [28] Loewner C. and Nirenberg L. *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, Orlando, FL (1974), 245272.
- [29] Marcus M. and Véron L., *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Annales de l'Institut Henri Poincard **14** (1997), 237-274.
- [30] Marcus M. and Véron L., *Trace au bord des solutions positives d'équations elliptiques et paraboliques non linéaires. Résultats d'existence et d'unicité*, [Boundary trace of positive solutions of nonlinear parabolic and elliptic equations. Existence and uniqueness results.] C. R. Acad. Sci. Paris Sér. I, Math. **323**, no.6 (1996) 603608.
- [31] Marcus M. and Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. rat. Mech. Anal. **144**, 201-231 (1998).
- [32] Marcus M. and Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. **77**, 481-524 (1998).
- [33] Marcus M. and Véron L., *Removable singularities and boundary trace*, J. Math. Pures Appl. **80**, 879-900 (2000).
- [34] Marcus M. and Véron L., *Capacitary estimates of positive solutions of semilinear elliptic equations with absorption*, J. European Math. Soc. **6**, 483-527 (2004).
- [35] Marcus M. and Véron L., *The precise boundary trace of positive solutions of the equation $\Delta u = u^q$ in the supercritical case*, in "Perspectives in Nonlinear Partial Differential Equations" (in honor of H. Brezis), Berestycki et al eds., Contemporary Mathematics A. M. S. **446** (2007) p. 345-384.

- [36] Marcus M. and Veron L., *On a characterisation of Besov spaces with negative exponents*, Topics Around the Research of Vladimir Mazya /International Mathematical Series 11-13, Springer Science+Business Media, 2009.
- [37] Marcus M. and Veron L., *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91** (2009) 256-295.
- [38] Mselati B., *Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation*, (English summary) Mem. Amer. Math. Soc. **168**, no. **798**, (2004) 121 pp.
- [39] Osserman R., *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. 7, 1641-1647 (1957).
- [40] Triebel H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Pub. Co., 1978.

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COMPLETE CLASSIFICATION OF THE POSITIVE SOLUTIONS OF $-\Delta u + u^q = 0$

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ABSTRACT. We study the equation $-\Delta u + u^q = 0$, $q > 1$, in a bounded C^2 domain $\Omega \subset \mathbb{R}^N$. A positive solution of the equation is *moderate* if it is dominated by a harmonic function and *σ -moderate* if it is the limit of an increasing sequence of moderate solutions. It is known that in the subcritical case, $1 < q < q_c = (N+1)/(N-1)$, every positive solution is σ -moderate [31]. More recently Dynkin proved, by probabilistic methods, that this remains valid in the supercritical case for $q \leq 2$, [15]. The question remained open for $q > 2$. In this paper we prove that, for all $q \geq q_c$, every positive solution is σ -moderate. We use purely analytic techniques which apply to the full supercritical range. The main tools come from linear and non-linear potential theory. Combined with previous results, this establishes a 1-1 correspondence between positive solutions and their boundary traces in the sense of [35].

1. INTRODUCTION

In this paper we study boundary value problems for the equation

$$(1.1) \quad -\Delta u + |u|^q \text{sign } u = 0, \quad q > 1$$

in a bounded C^2 domain Ω . We say that u is a solution of this equation if $u \in L^q_{\text{loc}}(\Omega)$ and the equation holds in the sense of distributions. Every solution of the equation is in $W^{2,\infty}_{\text{loc}}(\Omega)$. In particular, every solution is in $C^1(\Omega)$.

Let $\mathfrak{M}(\partial\Omega)$ denote the space of finite Borel measures on the boundary. Put

$$\rho(x) := \text{dist}(x, \partial\Omega)$$

and denote by $L^q_\rho(\Omega)$ the Lebesgue space with weight ρ .

For $\nu \in \mathfrak{M}(\partial\Omega)$ a (classical) weak solution of the boundary value problem

$$(1.2) \quad -\Delta u + |u|^q \text{sign } u = 0 \quad \text{in } \Omega, \quad u = \nu \quad \text{on } \partial\Omega$$

is a function $u \in L^1(\Omega) \cap L^q_\rho(\Omega)$ such that

$$(1.3) \quad -\int_\Omega u \Delta \phi \, dx + \int_\Omega |u|^q \text{sign } u \phi \, dx = -\int_{\partial\Omega} \partial_{\mathbf{n}} \phi \, d\nu,$$

for every $\phi \in C^2_0(\bar{\Omega})$ where

$$(1.4) \quad C^2_0(\bar{\Omega}) := \{\phi \in C^2(\bar{\Omega}) : \phi = 0 \quad \text{on } \partial\Omega\}.$$

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The boundary value problem (1.2) with data given by a finite Borel measure is well understood. It is known that, if a solution exists then it is unique. Gmira and Véron [20] proved that, if $1 < q < (N+1)/(N-1)$, the problem possesses a solution for every $\nu \in \mathfrak{M}(\partial\Omega)$; if $q \geq (N+1)/(N-1)$ then the problem has no solution for any measure ν concentrated at a point. The number $q_c := (N+1)/(N-1)$ is *the critical value* for (1.2). The interval $(1, (N+1)/(N-1))$ is the *subcritical* range; the interval $[(N+1)/(N-1), \infty)$ is the *supercritical* range.

In the early 90's the boundary value problem (1.2) became of great interest due to its relation to branching processes and superdiffusions (see Dynkin [11, 12], Le Gall [23]). At first, the study of the problem concentrated on the characterization of the family of finite measures for which (1.2) possesses a solution. This question is closely related to the characterization of removable boundary sets. A compact set $K \subset \partial\Omega$ is *removable* if every positive solution u of (1.1) which has a continuous extension to $\bar{\Omega} \setminus K$ can be extended to a function in $C(\bar{\Omega})$.

In a succession of works by Le Gall [24, 25] (for $q = 2$), Dynkin and Kuznetsov [16, 17] (for $1 < q \leq 2$) and Marcus and Véron [32, 33] (the first for $q \geq 2$, the second providing a new proof for all $q \geq q_c$) the following results were established.

Theorem A. *Let K be a compact subset of $\partial\Omega$. Then*

$$(1.5) \quad K \text{ is removable} \iff C^{2/q, q'}(K) = 0.$$

Here $q' = q/(q-1)$ and $C^{2/q, q'}$ denotes Bessel capacity on $\partial\Omega$.

Theorem B. *Let $\nu \in \mathfrak{M}(\partial\Omega)$. Problem (1.2) possesses a solution if and only if $\nu \prec C^{2/q, q'}$, i.e. ν vanishes on every Borel set $E \subset \partial\Omega$ such that $C^{2/q, q'}(E) = 0$.*

Remark A.1. For solutions in $L^q_\rho(\Omega)$, the removability criterion applies to signed solutions as well.

Remark A.2. For a non-negative solution u of (1.1), the removability criterion can be extended to an arbitrary set $E \subset \partial\Omega$. Suppose that u vanishes on every $C_{2/q, q'}$ -finely open subset of $\partial\Omega \setminus E$. Then

$$C^{2/q, q'}(E) = 0 \implies u = 0.$$

This is a consequence of the capacity estimates of [34].

In view of the estimates of Keller [22] and Osserman [39] equation (1.1) possesses solutions which are not in $L^q_\rho(\Omega)$. In particular the equation possesses solutions which blow up everywhere on the boundary (recall that we assume that Ω is of class C^2). Such solutions, called *large solutions* have been studied for a long time (see e.g. Loewner and Nirenberg [28] who studied the case $q = (N+2)/(N-2)$). It was established that the large solution is unique and its asymptotic behavior at the boundary was described (see Bandle and Marcus [7, 8] and the references therein). The uniqueness of

large solutions was also established for domains of class C^0 and even for $C_{2/q,q'}$ -finely open sets (see Marcus and Véron [29, 37]).

The next question in the study of equation (1.1) was whether it is possible to assign to arbitrary solutions a measure, not necessarily finite, which uniquely determines the solution. (Eventually such a measure was called a *boundary trace*.) In investigating this question, attention was restricted to positive solutions. The Herglotz theorem for positive harmonic functions served as a model. But, in contrast to the linear case, here one must allow unbounded measures.

In [24] Le Gall studied (1.1) with $q = 2$ and Ω a disk in \mathbb{R}^2 . He showed that, in this case, every positive solution possesses a boundary trace which uniquely determines the solution. The boundary trace was described in probabilistic terms and the proof relied mainly on probabilistic techniques.

In [30] Marcus and Véron introduced a notion of boundary trace (later Dynkin called it ‘the rough trace’) which can be described as a (possibly unbounded) Borel measure ν with the following properties. There exists a closed set $F \subset \partial\Omega$ such that

- (i) $\nu(E) = \infty$ for every non-empty Borel subset of F ,
- (ii) ν is a Radon measure on $\partial\Omega \setminus F$.

Let us denote the family of positive measures possessing these properties by $\mathcal{B}_{\text{reg}}(\partial\Omega)$. Given a positive solution u of (1.1), we say that it has (rough) boundary trace $\nu \in \mathcal{B}_{\text{reg}}(\partial\Omega)$ if (with F as above)

- (i') For every open neighborhood Q of F , $u \in L^1(\Omega \setminus A) \cap L^q_\rho(\Omega \setminus A)$ and (1.3) holds for every $\varphi \in C_0^2(\bar{\Omega})$ vanishing in a neighborhood of F .
- (ii') If $\xi \in F$ then, for every open neighborhood A of ξ ,

$$\int_{A \cap \Omega} u^q \rho \, dx = \infty.$$

The following result (announced in [30]) was proved in [31]:

Theorem C. *Every positive solution of (1.1) possesses a boundary trace in $\mathcal{B}_{\text{reg}}(\partial\Omega)$.*

If $1 < q < q_c$ then, for every $\nu \in \mathcal{B}_{\text{reg}}(\partial\Omega)$, (1.1) possesses a unique solution with boundary trace ν .

In the supercritical case it was shown in [32] that, under some additional conditions on ν , – mainly that ν must vanish on subsets of $\partial\Omega \setminus F$ of $C_{2/q,q'}$ -capacity zero, – (1.1) possesses a solution with rough trace ν . These conditions were shown to be necessary and sufficient for existence. However, it soon became apparent that in the supercritical case, the solution is no longer unique. A counterexample to this effect was constructed by Le Gall in 1997. Therefore, in order to deal with the super critical case, a more refined definition of boundary trace was necessary.

Kuznetsov [21] and Dynkin and Kuznetsov [18] provided such a definition, which they called ‘the fine trace’. Their definition was similar to that of the

rough trace, but the singular set F was not required to be closed in the Euclidean topology. Instead it was required to be closed with respect to a finer topology defined in probabilistic terms. With this definition they showed that, if $q \leq 2$ then, for any positive ‘fine trace’ ν , (1.1) possesses a solution the trace of which is equivalent, but not necessarily identical, to ν . The equivalence is defined in terms of polarity. Furthermore they showed that the minimal solution corresponding to a prescribed trace is σ -moderate and it is *the unique solution in this class*. The restriction to $q \leq 2$ is due to the fact that the proof was based on probabilistic techniques which do not apply to $q > 2$.

A σ -moderate solution was defined as the limit of an increasing sequence of positive moderate solutions. We recall that a moderate solution is a solution in $L^1(\Omega) \cap L^q_\rho(\Omega)$, i.e., a solution whose boundary trace is a finite measure.

In around the year 2002, Mselati proved in his Ph.D. thesis (under the supervision of Le Gall) that *for $q = 2$ every positive solution of (1.1) is σ -moderate*. This work appeared in [38]. Mselati used a combination of analytic and probabilistic techniques such as the ‘Brownian snake’ developed by Le Gall [27]. Following this, Dynkin [15] extended Mselati’s result proving:

If $q_c \leq q \leq 2$ then every positive solution of (1.1) is σ -moderate.

Instead of the ‘Brownian snake’ technique, which can be applied only to the case $q = 2$, Dynkin’s proof used new results on Markov processes that are applicable to $q \leq 2$.

At about the same time Marcus and Véron introduced a notion of boundary trace – they called it ‘the precise trace’ – based on the classical notion of $C^{2/q,q'}$ -fine topology (see [1]). A Borel measure ν on $\partial\Omega$ belongs to this family of traces, to be denoted by $\mathcal{F}^{2/q,q'}(\partial\Omega)$, if there exists a $C^{2/q,q'}$ -finely closed set $F \subset \partial\Omega$ such that:

- (i) $\nu(E) = \infty$ for every non-empty Borel subset of F .
- (ii) Every point $x \in \partial\Omega \setminus F$ has a $C_{2/q,q'}$ -finely open neighborhood Q_x such that $\nu(Q_x) < \infty$.
- (iii) If E is a Borel set such that $\nu(E) < \infty$ then ν vanishes on subsets of E of $C^{2/q,q'}$ -capacity zero.

In the subcritical case the $C^{2/q,q'}$ -fine topology is identical to the Euclidean topology and consequently the precise trace coincides with the rough trace.

With this definition they proved [35], by purely analytic methods:

Theorem D. *For every $q \geq q_c$:*

- (a) *Every positive solution of (1.1) possesses a boundary trace $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$.*
- (b) *For every measure $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$, problem (1.2) possesses a σ -moderate solution.*

(c) *The solution is unique in the class of σ -moderate solutions.*

The question whether every positive solution of (1.1) with $q > 2$ is σ -moderate remained open. In the present paper we settle this question proving,

Theorem 1 . *For every $q \geq q_c$, every positive solution of (1.1) is σ -moderate.*

The proof employs only analytic techniques and *applies to all $q \geq q_c$* . Of course the statement is also valid in the subcritical case, in which case it is an immediate consequence of Theorem C.

Combining Theorems C, D with Theorem 1 we obtain:

Corollary 1 . *For every $q > 1$ and every non-negative $\nu \in \mathcal{F}^{2/q,q'}(\partial\Omega)$, problem (1.2) possesses a unique solution. If $1 < q < q_c$, $\mathcal{F}^{2/q,q'}(\partial\Omega) = \mathcal{B}_{\text{reg}}(\partial\Omega)$.*

The method developed in the present paper can be adapted and applied to a general class of problems which includes boundary value problems for equations such as

$$-\Delta u + \rho^\alpha |u|^q \text{sign } u = 0, \quad \alpha > -2$$

and

$$-\Delta u + g(u) = 0,$$

where $g \in C(\mathbb{R})$ is odd, monotone increasing and satisfies the Δ_2 condition and the Keller–Osserman condition. For equations of the latter type, the method can be adapted to boundary value problems in Lipschitz domains as well. These results will be presented in a subsequent paper.

The main ingredients used in the present paper are:

(a) Nonlinear potential theory and fine topologies associated with Bessel capacities (see [1] and [35]).

(b) The theory of boundary value problems for equations of the form

$$L^V u := -\Delta u + Vu = 0 \quad \text{in } \Omega,$$

where $V > 0$ and $\rho^2 V$ is bounded. Here we use mainly the results of Ancona [3] together with classical potential theory results (see e.g. [2]).

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2. PRELIMINARIES: ON THE EQUATION $-\Delta u + Vu = 0$.

For the convenience of the reader we collect here some definitions and results of classical potential theory concerning operators of the form $L^V = -\Delta + V$, that will be used in the sequel. The results apply also to operators of the form $-L_0 + V$ where L_0 is a second order uniformly elliptic operator on

differentiable manifolds with negative curvature. However we shall confine ourselves to the operator L^V in a bounded domain $\Omega \subset \mathbb{R}^N$ which is either a C^2 domain or Lipschitz.

The following conditions on V will be assumed, without further mention, throughout the paper.

$$(2.1) \quad 0 < V \leq c\rho(x)^2, \quad V \in C(\Omega).$$

By [3] if Ω is a bounded Lipschitz domain, the Martin boundary can be identified with the Euclidean boundary $\partial\Omega$ and, for every $\zeta \in \partial\Omega$ there exists a positive L^V harmonic which vanishes on $\partial\Omega \setminus \{\zeta\}$. If normalized this harmonic is unique. We choose a fixed reference point, say $x_0 \in \Omega$ and denote by K_ζ^V this L^V harmonic, normalized by $K_\zeta^V(x_0) = 1$.

We observe that the positivity of V is essential for this result. Indeed the result depends on the weak coercivity of L^V (see definition in [3, Section 2]) which is guaranteed in our case by Hardy's inequality.

As a consequence of the above one obtains the following basic result (see Ancona [3], Theorem 3 and Corollary 13),

Representation Theorem. *For each positive L^V -harmonic function u in Ω there exists a unique positive measure μ on $\partial\Omega$ such that*

$$(2.2) \quad u(x) = \int_{\partial\Omega} K_\zeta^V d\mu(\zeta) \quad \forall x \in \Omega.$$

The function

$$K^V(\cdot, \zeta) = K_\zeta^V(\cdot)$$

is the Martin kernel. In C^2 -domains, with respect to a classical elliptic operator such as $-\Delta$, it can be identified with the Poisson kernel P . More precisely in this case

$$K^0(\cdot, \zeta) = P(\cdot, \zeta)/P(x_0, \zeta),$$

where x_0 is a fixed reference point in Ω . In Lip domains, with respect to $-\Delta$, K^0 is precisely the harmonic measure.

In the sequel we write

$$K_\zeta := K_\zeta^0.$$

The measure μ corresponding to an L^V harmonic u will be called *the L^V boundary trace of u* and we use the notation

$$(2.3) \quad \mathbb{K}_\mu^V := \int_{\partial\Omega} K_\zeta^V d\mu(\zeta), \quad \mathbb{K}_\mu := \int_{\partial\Omega} K_\zeta d\mu(\zeta).$$

Let D be a Lipschitz domain such that $\bar{D} \subset \Omega$ and $h \in L^1(\partial D)$. We denote by $S^V(D, h)$ the solution of the problem

$$(2.4) \quad L^V w := -\Delta w + Vw = 0 \text{ in } D, \quad w = h \text{ on } \partial D.$$

If μ is a finite measure on ∂D , $S^V(D, \mu)$ is defined in the same way. If D is a C^2 domain, a function $w \in L^1(D)$ is a solution of (2.4) (with h replaced

by μ) if

$$(2.5) \quad \int_D (-w\Delta\varphi + Vw\varphi)dx = - \int_{\partial D} \partial_{\mathbf{n}}\varphi d\mu,$$

for every $\varphi \in C_0^2(\bar{D})$.

A family of domains $\{\Omega_n\}$ such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\cup \Omega_n = \Omega$ is called an *exhaustion of Ω* . We say that $\{\Omega_n\}$ is a Lipschitz (resp. C^2) exhaustion if each domain Ω_n is Lipschitz (resp. C^2).

An l.s.c. function $u \in L_{\text{loc}}^1(\Omega)$ is L^V superharmonic if $L^V u \geq 0$ in distribution sense. Such a function is necessarily in $W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$ and consequently it possesses an L^1 trace on ∂D for every C^2 domain $D \Subset \Omega$. Furthermore, for every such domain, $u \geq S^V(D, u)$. If u is positive, the same holds for every Lipschitz domain $D \Subset \Omega$.

If u is an L^V -superharmonic in Ω and D a C^2 domain such that $D \Subset \Omega$ then the function u_D defined by

$$u_D = S^V(D, u) \text{ in } D, \quad u_D = u \text{ in } \Omega \setminus D$$

is called the *D-truncation of u* . This function is an L^V -superharmonic.

Lemma 2.1. *Let u be a non-negative L^V -superharmonic and $\{\Omega_n\}$ a Lipschitz exhaustion of Ω . Then the following limit exists*

$$(2.6) \quad \tilde{u} := \lim S^V(\Omega_n, u)$$

and \tilde{u} is the largest L^V -harmonic dominated by u .

Proof. The sequence $\{S^V(\Omega_n, u)\}$ is non-increasing. Consequently the limit exists and it is an L^V -harmonic. Every L^V harmonic v dominated by u must satisfy $v \leq S^V(\Omega_n, u)$ in Ω_n . Therefore \tilde{u} is the largest such harmonic. \square

Definition 2.2. *A non-negative L^V -superharmonic is called an L^V -potential if its largest L^V -harmonic minorant is zero.*

The following is an immediate consequence of Lemma 2.1.

Lemma 2.3. *A non-negative superharmonic function p is an L^V potential if and only if*

$$S^V(D_\beta, p) \rightarrow 0 \text{ as } \beta \rightarrow 0.$$

Riesz decomposition theorem. *Every non-negative L^V -superharmonic u can be written in a unique way in the form $u = p + h$ where p is an L^V potential and h a non-negative L^V -harmonic.*

Remark. In fact $h = \tilde{u}$ as defined in (2.6).

For further results concerning the L^V -potential see [2, Ch.I, sec. 4].

Definition 2.4. *Let $A \subset \Omega$ and let s be a positive L^V -superharmonic. Then R_s^A (called the reduction of s relative to A) is given by*

$$R_s^A = \text{lower envelope of } \{f : 0 \leq f \text{ superharmonic, } s \leq f \text{ on } A.\}$$

If A is open then R_s^A itself is L^V -superharmonic so that the lower envelope is simply the minimum, [2, p.13].

Definition 2.5. Let $\zeta \in \partial\Omega$. A set A is L^V thin at ζ (in French 'A est ζ -effilé') if $R_{K_\zeta^V}^A \not\equiv K_\zeta^V$.

In view of a theorem of Brelot, if A is open:

$$R_{K_\zeta^V}^A \not\equiv K_\zeta^V \iff R_{K_\zeta^V}^A \text{ is an } L^V\text{-potential.}$$

Furthermore, even if A is not open there exists an open set O such that $A \subset O$ and O is thin at ζ .

Lemma 2.6. Assume that A is thin at $\zeta \in \partial\Omega$ and A open. Let $\{D_n\}$ be a C^2 exhaustion of Ω and put $A_n = \partial\Omega_n \cap A$. Then

$$S^V(\Omega_n, K_\zeta^V \chi_{A_n}) \rightarrow 0.$$

Definition 2.7. Let $\zeta \in \partial\Omega$ and f a real function on Ω . We say that f admits the fine limit ℓ at ζ if there exists a closed set $E \subset \Omega$ such that E is thin at ζ and

$$\lim_{x \rightarrow \zeta, x \in \Omega \setminus E} f(x) = \ell.$$

To indicate this type of convergence we write,

$$\lim_{x \rightarrow \zeta} f(x) = \ell, \text{ } L^V\text{-finely.}$$

Recall that there also exists an open set A such that $E \subset A$ and A is thin at ζ .

Proposition 2.8. Let u be a positive L^V harmonic function, or a solution of (1.1). For $\zeta \in \partial\Omega$,

$$\lim_{x \rightarrow \zeta} u = b \quad L^V\text{-finely} \implies \lim_{x \rightarrow \zeta} u = b \text{ n.t.,}$$

where 'n.t.' means 'non-tangentially'.

Proof. Let $\rho(x) := \text{dist}(x, \partial\Omega)$. By [2, Lemma 6.4], if A is an L^V thin set at ζ and $\beta_n \downarrow 0$ then

$$A \cap \{x \in \Omega : |x - \zeta| < \rho(x), \beta_n/2 < \rho(x) < 3/2\beta_n\} \neq \emptyset$$

for all sufficiently large n . Therefore the assertion follows from Harnack's inequality. \square

For the next two theorems see [2, Prop.1.6 & Thm. 1.8].

Theorem 2.9. If p is an L^V potential then, for every positive L^V harmonic v :

$$\lim_{x \rightarrow \zeta, \text{fine}} p/v = 0 \quad \mu_v\text{-a.e.}$$

where μ_v is the L^V boundary trace of v .

Theorem 2.10. [Fatou-Doob-Naim] *If u, v are positive L^V harmonics then u/v admits a fine limit μ_v a.e. Furthermore*

$$\lim_{x \rightarrow \zeta, \text{fine}} u/v = f = \frac{d\mu_u}{d\mu_v} \quad \mu_v - \text{a.e.}$$

where μ_u and μ_v are the L^V boundary traces of u and v respectively and the term on the right hand side denotes the Radon-Nikodym derivative.

The next lemma – an application of the theorem of Fatou – is due to Ancona [5].

Lemma 2.11. *Assume that v is a positive L^V harmonic function with L^V boundary trace ν . Then*

$$(2.7) \quad \lim_{x \rightarrow \zeta} v > 0 \quad \text{n.t. } \nu\text{-a.e. } \zeta \in \partial\Omega.$$

If $\nu \perp \mathbb{H}_{N-1}$ then

$$(2.8) \quad \lim_{x \rightarrow \zeta} v = \infty \quad \text{n.t. } \nu\text{-a.e.}$$

Proof. The function 1 is an L^V superharmonic. If it is a potential then, by Theorem 2.9

$$\lim_{x \rightarrow \zeta} 1/v = 0 \quad L^V\text{-finely } \nu\text{-a.e.}$$

Therefore, by Proposition 2.8

$$\lim_{x \rightarrow \zeta} v = \infty \quad \text{n.t. } \nu\text{-a.e.}$$

If 1 is not a potential there exists a positive L^V harmonic w and a potential p such that $1 = w + p$. Let $w = K_\gamma^V$ and put $d\gamma/d\nu =: f$. By Theorem 2.10

$$\lim w/v = f \quad L^V\text{-finely } \nu\text{-a.e.}$$

(We do not exclude the possibility that $f = 0$ ν -a.e. but, of course, $f < \infty$ ν -a.e.) Since $p/v \rightarrow 0$ finely ν -a.e., it follows that

$$\lim 1/v = \lim(w + p)/v = f \quad L^V\text{-finely } \nu\text{-a.e.}$$

Applying again Proposition 2.8 we obtain

$$\lim 1/v = f \quad \text{n.t. } \nu\text{-a.e.}$$

which in turn implies (2.7).

If $\nu \perp \mathbb{H}_{N-1}$ then $f = 0$ ν -a.e. and consequently $v \rightarrow \infty$ n.t. ν -a.e. □

3. MODERATE SOLUTIONS OF $L^V u = 0$

We recall some definitions from [15] following the notation of [6].

Definition 3.1. We shall say that a boundary point ζ is L^V regular if $\tilde{K}^V(\cdot, \zeta)$ ($=$ the largest L^V harmonic dominated by the L^V superharmonic $K(\cdot, \zeta)$) is positive. The point ζ is L^V singular if $\tilde{K}^V(\cdot, \zeta) = 0$, i.e. $K(\cdot, \zeta)$ is an L^V potential.

We denote by $Sing(V)$, respectively $Reg(V)$, the set of singular points, respectively regular points, of L^V .

Remark. If ζ is L^V -regular then

$$\tilde{K}^V(\cdot, \zeta) = c(x_0)K^V(\cdot, \zeta), \quad c(x_0) = \tilde{K}^V(x_0, \zeta),$$

where x_0 is a fixed reference point in Ω such that

$$K^V(x_0, \zeta) = 1 \quad \forall \zeta \in \partial\Omega.$$

Notation. The family of finite Borel measures on a set A is denoted by $\mathfrak{M}(A)$. For $A = \partial\Omega$ we shall write simply \mathfrak{M} . If $\mu \in \mathfrak{M}(A)$ we denote by $|\mu|$ the total variation measure and by $\|\mu\|_{\mathfrak{M}(A)}$ the total variation norm.

Definition 3.2. An L^V harmonic u is L^V -moderate if $u \in L^1(\Omega) \cap L^1(\Omega; V\rho)$ and there exists a measure $\nu \in \mathfrak{M}$ such that

$$(3.1) \quad \int_{\Omega} (-u\Delta\varphi + uV\varphi)dx = - \int_{\partial\Omega} \partial_{\mathbf{n}}\varphi d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$.

A measure $\nu \in \mathfrak{M}$ is L^V -moderate if there exists a moderate L^V harmonic satisfying (3.1).

The space of L^V moderate measures is denoted by \mathfrak{M}^V .

Definition 3.3. Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$. We say that u possesses an m -boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$ if, for every C^2 -exhaustion of Ω , say $\{\Omega_n\}$,

$$u|_{\partial\Omega_n} d\mathbb{H}_{N_1} \rightharpoonup \nu,$$

weakly with respect to $C(\bar{\Omega})$, i.e.,

$$(3.2) \quad \int_{\partial\Omega_n} uh dS \rightarrow \int_{\partial\Omega} h d\nu \quad \forall h \in C(\bar{\Omega}).$$

If ν is the m -boundary trace of u we write $\text{tr } u := \nu$.

Remark. If u possesses an m -boundary trace ν then $u \in L^1(\Omega)$ and

$$(3.3) \quad \sup \int_{\partial\Omega_n} |u| dS < \infty.$$

This follows immediately from the definition. It is easily verified that, if u is L^V moderate and satisfies (3.2) then ν is the m -boundary trace of u .

Notation. Let $\rho(x) := \text{dist}(x, \partial\Omega)$. In the case of C^2 domains, there exists $\beta_0 > 0$ such that for $x \in \Omega$, $\rho(x) < \beta_0$, there exists a unique point on $\partial\Omega$, to be denoted by $\sigma(x)$, such that

$$|x - \sigma(x)| = \rho(x).$$

Thus

$$x - \sigma(x) = \rho(x)\nu_{\sigma(x)}$$

where ν_ζ denotes the unit normal at $\zeta \in \partial\Omega$ pointing into the domain. (We also denote $-\nu_\zeta =: \mathbf{n}_\zeta$.) It can be shown that the function $x \mapsto \sigma(x)$ is in $C^2(\bar{\Omega}^0)$ where

$$\Omega^0 := \{x \in \Omega : 0 < \rho(x) < \beta_0\}.$$

The mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^2 homeomorphism of Ω^0 onto

$$\{(\rho, \sigma) \in \mathbb{R}_+ \times \partial\Omega : 0 < \rho < \beta_0\}.$$

Thus (ρ, σ) can be used as an alternative set of coordinates in Ω^0 ; we call them ‘flow coordinates’.

Put,

$$D_\beta = [x \in \Omega, \rho(x) > \beta], \quad \Omega_\beta = \Omega - \bar{D}_\beta, \quad \Sigma_\beta = [x \in \Omega, \rho(x) = \beta].$$

and for $\alpha \in (0, \infty)$ and $\zeta \in \partial\Omega$

$$\mathcal{C}_\zeta^\alpha = \{x \in \Omega^0 : |x - \zeta| > \alpha\rho(x)\}.$$

When $\alpha = 1$, the upper index will be omitted.

In the sequel we assume that Ω is a bounded C^2 domain.

Lemma 3.4. (i) If v is a positive L^V superharmonic and

$$(3.4) \quad \int_{\Omega} vV\rho \, dx < \infty.$$

then v is moderate. In particular, $v \in L^1(\Omega)$ and it possesses an m -boundary trace $\nu \in \mathfrak{M}$. The supremum of L^V -harmonics dominated by v , say v' , is an L^V harmonic and has the same m -boundary trace.

(ii) If v is a positive L^V superharmonic and v possesses an m -boundary trace $\nu \in \mathfrak{M}$ then v' , the supremum of L^V -harmonics dominated by v , is an L^V moderate harmonic and $\text{tr } v' \leq \nu$. If v is not a potential then v' is positive.

(iii) If v is an L^V harmonic (not necessarily positive) and v possesses an m -boundary trace ν then v is L^V moderate.

Proof. (i) Let $w \in L^1(\Omega)$ be the (unique) solution of the problem

$$(3.5) \quad -\Delta w = Vv \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

The solution exists because v satisfies (3.4). Then

$$-\Delta(w + v) \geq 0$$

and consequently $w + v \in L^1(\Omega)$ and it possesses an m -boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$. As $w \in L^1(\Omega)$ and has m -boundary trace zero, it follows that $v \in L^1(\Omega)$ and has m -boundary trace ν .

Given $\varphi \in C_0^2(\bar{\Omega})$, for each $\beta \in (0, \beta_0/2)$ we can construct a function $\varphi_\beta \in C_0^2(\bar{D}_\beta)$ such that

$$\varphi_\beta(x) = \varphi(\rho(x) - \beta, \sigma(x)) \text{ for } \beta \leq \rho(x) < \beta + \beta_0/4$$

and

$$\varphi_\beta \rightarrow \varphi \text{ in } C^2(\Omega), \quad \sup_\beta \|\varphi_\beta\|_{C^2(\bar{D}_\beta)} < \infty.$$

Let

$$v_\beta = S^V(D_\beta, v).$$

Then $0 \leq v_\beta \leq v$ and $v_\beta \downarrow v'$ as $\beta \downarrow 0$. Furthermore

$$\int_{D_\beta} (-v_\beta \Delta \varphi_\beta + v_\beta V \varphi_\beta) dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}} \varphi_\beta v dS.$$

Since $v \in L^1(\Omega; V\rho) \cap L^1(\Omega)$ and v possesses m- boundary trace ν on $\partial\Omega$ we obtain (by going to the limit as $\beta \rightarrow 0$):

$$(3.6) \quad \int_{\Omega} (-v' \Delta \varphi + v' V \varphi) dx = - \int_{\partial\Omega} \partial_{\mathbf{n}} \varphi d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$.

(ii) If v is a positive L^V superharmonic then there exists a Radon measure $\lambda > 0$ in Ω such that $L^V u = \lambda$ in the sense of distributions. Therefore $v \in W_{\text{loc}}^{1,p}(\Omega)$ for some $p > 1$. Let $v_\beta = S^V(D_\beta, v)$. Then $v_\beta \leq v$ in D_β and

$$(3.7) \quad \int_{D_\beta} (-v_\beta \Delta \varphi + v_\beta V \varphi) dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}} \varphi v dS$$

for every $\varphi \in C^2(\bar{D}_\beta)$. Choosing φ to be the solution of

$$-\Delta \varphi = 1 \text{ in } D_\beta, \quad \varphi = 0 \text{ on } \partial D_\beta$$

we obtain

$$(3.8) \quad \|v_\beta\|_{L^1(D_\beta)} + \|v_\beta\|_{L^1(D_\beta; V\rho_\beta)} \leq C \int_{\partial D_\beta} v dS$$

with a constant C independent of β . Here ρ_β is the first eigenfunction of $-\Delta$ in D_β normalized by $\rho_\beta(x_0) = 1$. Since, by assumption, v has an m-boundary trace, the right hand side of the inequality is bounded. In addition, ρ_β tends to the first normalized eigenfunction of $-\Delta$ in Ω . Therefore $v_\beta \downarrow v'$ as $\beta \downarrow 0$, locally uniformly in Ω and $v' \in L^1(\Omega) \cap L^1(\Omega; V\rho)$. By (3.7) with $\varphi = \varphi_\beta$ and Fatou's lemma we obtain – using the fact that $v_\beta \leq v \in L^1(\Omega)$ and $\varphi_\beta \rightarrow \varphi$ in $C^2(\Omega)$ –

$$(3.9) \quad \int_{\Omega} (-v' \Delta \varphi + v' V \varphi) dx \leq - \int_{\partial\Omega} \partial_{\mathbf{n}} \varphi d\nu,$$

for every non-negative $\varphi \in C_0^2(\bar{\Omega})$. Consequently (by a standard argument) v' has an m-boundary trace, say ν' , such that $\nu' \leq \nu$.

If, in addition, v is not an L^V potential then $v' > 0$.

(iii) The proof is essentially the same as that of part (ii) except that, in the present case, inequality (3.8) is replaced by

$$(3.10) \quad \|v_\beta\|_{L^1(D_\beta)} + \|v_\beta\|_{L^1(D_\beta; V\rho_\beta)} \leq C \int_{\partial D_\beta} |v| dS.$$

This inequality is proved by a standard argument as in e.g. [32]. Since v is an L^V harmonic, $v_\beta = v$ in D_β . Therefore we obtain $v \in L^1(\Omega) \cap L^1(\Omega; V\rho)$ and

$$\int_{D_\beta} (-v\Delta\varphi_\beta + vV\varphi_\beta)dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}}\varphi_\beta v dS.$$

Finally, taking the limit as $\beta \rightarrow 0$, we obtain

$$\int_{\Omega} (-v\Delta\varphi + vV\varphi)dx = - \int_{\partial D_\beta} \partial_{\mathbf{n}}\varphi_\beta d\nu.$$

□

Lemma 3.5. (i) If $\nu \in \mathfrak{M}^V$ then the solution of (3.1) is unique. It will be denoted by \mathbb{M}_ν^V .

(ii) The space \mathfrak{M}^V is linear and

$$(3.11) \quad 0 \leq \nu \iff 0 \leq \mathbb{M}_\nu^V$$

(iii) Let $\tau \in \mathfrak{M}$ and $\nu \in \mathfrak{M}^V$.

$$(3.12) \quad |\tau| \leq \nu \implies \tau \in \mathfrak{M}^V,$$

$$(3.13) \quad \nu \in \mathfrak{M}^V \implies |\nu| \in \mathfrak{M}^V.$$

(iv) If $\nu \in \mathfrak{M}^V$ then $|\mathbb{M}_\nu^V| \leq \mathbb{M}_{|\nu|}^V$.

Proof. (i), (ii) and (3.12) are classical. We turn to the proof of (3.13). Put $u = \mathbb{M}_\nu^V$. Since $uV \in L_\rho^1(\Omega)$, there exist v^+ and v^- in $L^1(\Omega) \cap L^1(\Omega; V\rho)$ such that

$$-\Delta v^\pm + Vu_\pm = 0 \text{ in } \Omega, \quad v^\pm = \nu_\pm \text{ on } \partial\Omega.$$

It follows that

$$u = v^+ - v^-, \quad |u| \leq v^+ + v^- =: w$$

and

$$-\Delta w + Vw \geq -\Delta w + V|u| = 0.$$

Thus w is a positive L^V superharmonic with m-boundary trace $|\nu|$ and

$$w \in L^1(\Omega) \cap L^1(\Omega; V\rho).$$

By Lemma 3.4 (i), the largest L^V harmonic dominated by w , say w' , is L^V moderate and

$$\text{tr } w' = \text{tr } w = |\nu|.$$

Thus $w' = \mathbb{M}_{|\nu|}$. Since $0 < w'$ and $u < w'$ it follows that $|u| \leq w'$, which is precisely assertion (iv). □

Denote,

$$(3.14) \quad \mathfrak{M}_0^V := \{\nu \in \mathfrak{M} : \mathbb{K}_{|\nu|}V \in L_\rho^1(\Omega)\}.$$

The following is an immediate consequence of Lemma 3.4:

Lemma 3.6. $\mathfrak{M}_0^V \subset \mathfrak{M}^V$.

Proof. If $\nu \in \mathfrak{M}_0^V$ then $\mathbb{K}_{|\nu|}$ is an L^V superharmonic satisfying (3.4). □

Remark. In general there may exist positive measures ν such that $\mathbb{K}_\nu^V \in L^1(\Omega; V\rho)$ but $\mathbb{K}_\nu \notin L^1(\Omega; V\rho)$.

Lemma 3.7. *If $V \in L_\rho^{q'}(\Omega)$ for some $q' > 1$ then every positive measure in $W^{-2/q,q}$ belongs to \mathfrak{M}_0^V .*

Proof. If $\nu \in W^{-2/q,q}$ then $\mathbb{K}_\nu \in L_\rho^q(\Omega)$ (see [40, 1.14.4.] or [36]). Therefore $V\mathbb{K}_\nu \in L_\rho^1(\Omega)$. If, in addition, $\nu \geq 0$ then $\nu \in \mathfrak{M}_0^V$. \square

Remark. There are signed measures $\nu \in W^{-2/q,q}$ such that $|\nu| \notin W^{-2/q,q}$. Therefore in general $W^{-2/q,q}$ may not be contained in \mathfrak{M}_0^V .

Proposition 3.8. *Let v be a positive, L^V moderate harmonic with m-boundary trace ν . Let ν' be the L^V boundary trace of v , i.e.,*

$$(3.15) \quad \mathbb{M}_\nu^V = \mathbb{K}_{\nu'}^V.$$

Then

$$(3.16) \quad \nu'(E) = 0 \iff \nu(E) = 0.$$

Furthermore,

$$(3.17) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \quad \nu' - a.e.$$

Let $F \subset \partial\Omega$ be compact and denote

$$v_F := \inf(v, \mathbb{K}_{\nu_F}), \quad \nu_F := \nu\chi_F.$$

Then v_F is a moderate supersolution of L^V and the largest L^V harmonic dominated by v_F is given by

$$(3.18) \quad (v_F)' = \mathbb{K}_{\nu'_F}^V.$$

Proof. Since $vV \in L_\rho^1(\Omega)$ it follows (by Fubini) that

$$\int_{\partial\Omega} \left(\int_{\Omega} K^V(\cdot, \zeta)V\rho \right) d\nu'(\zeta) < \infty$$

which implies (3.17).

Let F be a compact subset of $\partial\Omega$. If $\nu'(F) > 0$ then $K_{\nu'_F}^V$ (as usual $\nu'_F := \nu'\chi_F$) is a positive L^V moderate harmonic which vanishes on $\partial\Omega \setminus F$ and is dominated by v . Therefore

$$K_{\nu'_F}^V \leq v_F,$$

and consequently $\nu(F) > 0$.

Next we show that $\nu(F) > 0$ implies $\nu'(F) > 0$. Since $\nu(F) > 0$, v_F is a positive L^V superharmonic with m-boundary trace ν_F . In addition $v_F \leq v \in L^1(\Omega; V\rho)$. Therefore by Lemma 3.4 (i), if $(v_F)'$ is the largest L^V harmonic dominated by v_F then $(v_F)'$ is L^V moderate with m-boundary trace ν_F . Thus

$$0 < (v_F)' \leq v_F.$$

On the other hand, the largest L^V harmonic dominated by v and vanishing on $\partial\Omega \setminus F$ is $\mathbb{K}_{\nu'_{\chi_F}}^V$. It follows that

$$0 < (v_F)' = \mathbb{K}_{\nu'_{\chi_F}}^V$$

which implies that $\nu'(F) > 0$. \square

Proposition 3.9. (i) For every $\zeta \in \partial\Omega$,

$$(3.19) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \iff \zeta \text{ is } L^V\text{-regular.}$$

(ii) If ν is a positive measure in \mathfrak{M}^V then $K^V(\cdot, \zeta)$ is L^V -moderate ν -a.e. and

$$(3.20) \quad \nu(\text{Sing}(V)) = 0.$$

(iii) If $V \in L^{q'}(\omega)$ for some $q' > 1$ then $C_{2/q, q'}$ -a.e. point $\zeta \in \partial\Omega$ is L^V regular. (Here $\frac{1}{q} + \frac{1}{q'} = 1$.)

Proof. (i) Assume that $K^V(\cdot, \zeta)V \in L_\rho^1(\Omega)$. Then, by Lemma 3.4, K_ζ^V is L^V moderate. Its m-boundary trace $\tau_\zeta \in \mathfrak{M}$ is concentrated at ζ . Thus $\tau_\zeta = a(\zeta)\delta_\zeta$ for some $a > 0$. It follows that K_ζ^V is a subsolution of the boundary value problem

$$-\Delta z = 0 \text{ in } \Omega, \quad z = \tau_\zeta \text{ on } \partial\Omega.$$

Therefore

$$K^V(\cdot, \zeta) \leq a(\zeta)K(\cdot, \zeta).$$

This implies that $\tilde{K}_\zeta^V > 0$, i.e., ζ is regular.

Assume that ζ is L^V regular. Then, by definition, $K(\cdot, \zeta)$ is not a potential and has m-boundary trace $= \delta_\zeta$. By Lemma 3.4(ii) the largest L^V harmonic dominated by $K(\cdot, \zeta)$, which we denote by $\tilde{K}^V(\cdot, \zeta)$, is L^V moderate and its boundary trace, say λ , is a positive measure bounded by δ_ζ . By uniqueness of the positive, normalized L^V harmonic vanishing on $\partial\Omega \setminus \{\zeta\}$,

$$K^V(\cdot, \zeta) = \tilde{K}^V(\cdot, \zeta)/\tilde{K}^V(x_0, \zeta).$$

Thus $K^V(\cdot, \zeta)V \in L_\rho^1(\Omega)$.

(ii) By (3.17) and (3.16),

$$(3.21) \quad K^V(\cdot, \zeta)V \in L_\rho^1(\Omega) \quad \nu - a.e.$$

By (i), this implies the second assertion.

(iii) In this case, every positive measure $\nu \in W^{-2/q, q}(\partial\Omega)$ is in \mathfrak{M}_0^V which is contained in \mathfrak{M}^V . It follows that the set of singular points of L^V must have $C_{2/q, q'}$ -capacity zero. \square

4. PRELIMINARIES: ON THE EQUATION $-\Delta u + u^q = 0$

In this section we collect some definitions and known results on positive solutions of (1.1), that will be needed for the proof of the main result.

A basic concept in this theory is that of $C_{2/q,q'}$ -fine topology. For the general theory of $C_{m,p}$ capacity and $C_{m,p}$ -fine topology we refer the reader to [1]. For more special results, useful in our study, we refer the reader to the summary in [35, Section 2].

The closure of a set $A \subset \partial\Omega$ in $C_{2/q,q'}$ -fine topology will be denoted by \tilde{A} . We shall say that two sets A, B are $C_{2/q,q'}$ equivalent (or briefly q-equivalent) if $C_{2/q,q'}(A \Delta B) = 0$.

There exists a constant c such that for every set A

$$C_{2/q,q'}(\tilde{A}) \leq cC_{2/q,q'}(A).$$

We recall the definition of regular and singular boundary points of a positive solution u of (1.1). A point $\zeta \in \partial\Omega$ is a q -regular point of u if there exists a $C_{2/q,q'}$ neighborhood of ζ , say O_ζ such that

$$\int_{O_\zeta \cap \Omega} u^q \rho dx < \infty.$$

ζ is q-singular if it is not q-regular. The set of q-regular points is denoted by $\mathcal{R}(u)$ and the set of singular points by $\mathcal{S}(u)$. Evidently $\mathcal{R}(u)$ is $C_{2/q,q'}$ open.

If F is a $C_{2/q,q'}$ -finely closed subset of $\partial\Omega$ then there exists an increasing sequence of compact subsets $\{F_n\}$ such that $C_{2/q,q'}(F \setminus F_n) \rightarrow 0$.

If u is a positive solution of (1.1) we say that it vanishes on a $C_{2/q,q'}$ -finely open set $O = \partial\Omega \setminus F$ if it is the limit of an increasing sequence of positive solutions $\{u_n\}$ such that $u_n \in C(\bar{\Omega} \setminus F_n)$ and $u_n = 0$ on $\partial\Omega \setminus F_n$. The q-support of the boundary trace of u – denoted by $\text{supp}_{\partial\Omega}^q u$ – is the complement of the largest $C_{2/q,q'}$ -finely open subset of $\partial\Omega$ where u vanishes.

Let $\nu \in \mathfrak{M}$. We say that u is a solution of the problem

$$(4.1) \quad -\Delta u + |u|^q \text{sign } u = 0 \text{ in } D, u = \nu \text{ on } \partial D$$

if $u \in L^1(\Omega) \cap L_\rho^q(\Omega)$ and

$$(4.2) \quad -\int_\Omega u \Delta \varphi dx + \int_\Omega |u|^q \text{sign } u \varphi dx = -\int_\partial \Omega \partial_n \varphi d\nu,$$

for every $\varphi \in C_0^2(\bar{\Omega})$. If a solution exists it is unique; it will be denoted by u_ν . If ν is a measure for which a solution exists, we say that it is q-good. The family of q-good measures is denoted by \mathcal{G}_q . It is known that [9] ν is q-good if and only if it vanishes on sets of $C_{2/q,q'}$ capacity zero. Furthermore, a positive measure $\nu \in \mathfrak{M}$ is q-good if and only if it is the limit of an increasing bounded sequence of measures in $W^{-2/q,q}$. In particular a measure $\nu \in \mathfrak{M}$ such that $|\nu| \in W^{-2/q,q}$ is a q-good measure.

A solution u of (1.1) is *moderate* if $u \in L^1(\Omega) \cap L_\rho^q(\Omega)$. A moderate solution possesses a boundary trace $\nu \in \mathfrak{M}$ such that (4.2) holds.

Denote by \mathcal{U}_q the set of positive solutions of (1.1). A solution $u \in \mathcal{U}_q$ is σ -moderate if it is the limit of an increasing sequence of moderate solutions.

A compact set $F \subset \partial\Omega$ is q -removable if a non-negative solution of (1.1) vanishing on $\partial\Omega \setminus F$ must vanish in Ω . An arbitrary set $A \subset \partial\Omega$ is q -removable if every compact subset is q -removable. It is known that A is q -removable if and only if $C_{2/q,q'}(A) = 0$ (see [33] and the references therein).

By [34], if $\{u_n\}$ is a sequence of positive solutions of (1.1) then

$$(4.3) \quad C_{2/q,q'}(\text{supp}_{\partial\Omega}^q u_n) \rightarrow 0 \implies u_n \rightarrow 0 \text{ locally uniformly in } \Omega.$$

If F is a $C_{2/q,q'}$ -finely closed subset of $\partial\Omega$, denote

$$U_F = \sup\{u \in \mathcal{U}_q : \text{supp}_{\partial\Omega}^q u \subset F\}.$$

It is well known that U_F is a solution of (1.1) and it vanishes on $\partial\Omega \setminus F$. We call it *the maximal solution relative to F*.

For an arbitrary Borel set $A \subset \partial\Omega$ denote

$$W_A = \sup\{u_\nu : \nu \in W^{-2/q,q}, \nu(\partial\Omega \setminus A) = 0\}$$

It is proved in [34] that

$$W_A = W_{\bar{A}}$$

and, if F is $C_{2/q,q'}$ -finely closed,

$$U_F = W_F.$$

In particular U_F is σ -moderate.

If v is a positive supersolution of (1.1) then the set of solutions dominated by it contains a maximal solution:

$$v^\# := \sup\{u \in \mathcal{U}_q : u \leq v\} \in \mathcal{U}_q.$$

If v is a positive subsolution of (1.1) then the set of solutions dominating it is non-empty and contains a minimal solution:

$$v_\# := \inf\{u \in \mathcal{U}_q : u \geq v\} \in \mathcal{U}_q.$$

If $u, v \in \mathcal{U}_q$ then $u + v$ is a supersolution, $(u - v)_+$ is a subsolution and we denote

$$u \oplus v = [u + v]^\#, \quad u \ominus v = [(u - v)_+]^\#.$$

If $u \in \mathcal{U}_q$ and F is a $C_{2/q,q'}$ -finely closed subset of $\partial\Omega$ we denote:

$$[u]_F = \inf(u, U_F)^\#.$$

If D is a C^2 subdomain of Ω and $h \in L^1(\partial D)$ we denote by $S_q(D, h)$ the solution of the problem

$$-\Delta u + |u|^q \text{sign } u = 0 \text{ in } D, u = h \text{ on } \partial D.$$

Let $\{\Omega_n\}$ be a C^2 exhaustion of Ω . Then, if v is a positive supersolution,

$$S_q(\Omega_n, v) \downarrow v^\#$$

and if v is a positive subsolution

$$S_q(\Omega_n, v) \uparrow v_\#.$$

The following definitions were introduced in [35]. A positive Borel measure τ on $\partial\Omega$ (not necessarily bounded) is called a *perfect measure* if it satisfies the following conditions:

(a) τ is outer regular relative to $C_{2/q,q'}$ -fine topology, i.e., for every Borel set E ,

$$\tau(E) = \inf\{\tau(Q) : Q \text{ is } C_{2/q,q'}\text{-finely open, } E \subset Q\}.$$

(b) If Q is a $C_{2/q,q'}$ -finely open set and A a Borel set such that $C_{2/q,q'}(A) = 0$ then $\tau(Q) = \tau(Q \setminus A)$.

The space of perfect measures is denoted by \mathcal{M}_q .

We observe that (b) implies:

(b') If Q is a $C_{2/q,q'}$ -finely open set, A a Borel subset such that $C_{2/q,q'}(A) = 0$ and $\tau(Q \setminus A) < \infty$ then $\tau(A) = 0$ and $\tau\chi_Q$ is a q -good measure.

For $\tau \in \mathcal{M}_q$ put

$$Q_\tau = \bigcup\{Q : Q \text{ is } C_{2/q,q'}\text{-finely open, } \tau(\tilde{Q}) < \infty\}.$$

If $u \in \mathcal{U}_q$ we say that u has boundary trace $\tau \in \mathcal{M}_q$ if:

(i) $\mathcal{R}(u) = Q_\tau$ and

(ii) for every $\xi \in Q_\tau$ there exists a $C_{2/q,q'}$ -finely open neighborhood Q such that $[u]_{\tilde{Q}}$ is a moderate solution with boundary trace $\tau\chi_{\tilde{Q}}$.

The boundary trace of u in this sense is called *the precise trace* and is denoted by $\text{tr } u$.

By [35, Theorem 5.11], for every $u \in \mathcal{U}_q$, there exists a sequence $\{Q_n\}$ of $C_{2/q,q'}$ -finely open subsets of $\mathcal{R}(u)$ such that

$$\tilde{Q}_n \subset Q_{n+1}, \quad [u]_{\tilde{Q}_n} \text{ is moderate } \quad \forall n, \quad C_{2/q,q'}(\mathcal{R}(u) \setminus \bigcup_n Q_n) = 0.$$

Such a sequence is called a *regular decomposition* of $\mathcal{R}(u)$. We denote:

$$(4.4) \quad \begin{aligned} \mathcal{R}_0(u) &= \bigcup_n Q_n, \quad \nu_n = \text{tr } [u]_{\tilde{Q}_n}, \\ u_{\mathcal{R}} &= \lim [u]_{\tilde{Q}_n}, \quad \nu_{\mathcal{R}} = \lim \nu_n. \end{aligned}$$

$u_{\mathcal{R}}$ and $\nu_{\mathcal{R}}$ do not depend on the specific sequence $\{Q_n\}$. In fact (by the theorem cited above)

$$(4.5) \quad [u]_F = [u_{\mathcal{R}}]_F \quad \forall F \text{ } C_{2/q,q'}\text{-finely closed, } F \subset \mathcal{R}(u),$$

and $u \ominus u_{\mathcal{R}}$ vanishes on $\mathcal{R}(u)$.

The following result is proved in [35] (see Theorem 5.16 and the remark following it):

Theorem 4.1. *Every positive solution u of (1.1) possesses a boundary trace $\nu \in \mathcal{B}_q$. Conversely, for every $\nu \in \mathcal{B}_q$ there exists a solution of (1.1) with*

boundary trace ν . Furthermore there exists a unique σ -moderate solution u of (1.1) with $\text{tr } u = \nu$, namely,

$$u = u_{\mathcal{R}} \oplus U_{\mathcal{S}(u)}.$$

where $u_{\mathcal{R}}$ is the σ -moderate solution defined in (4.4).

In addition, by [35, Theorem 5.11] we obtain:

Theorem 4.2. *If $u \in \mathcal{U}_q$ then*

$$(4.6) \quad \max(u_{\mathcal{R}}, [u]_{\mathcal{S}}) \leq u \leq u_{\mathcal{R}} + [u]_{\mathcal{S}}.$$

Proof. $v := u \ominus u_{\mathcal{R}}$ vanishes on $\mathcal{R}(u)$, i.e., $\text{supp}_{\partial\Omega}^q v \subset \mathcal{S}(u)$. Thus v is a solution dominated by u and supported in $\mathcal{S}(u)$, which implies that $v \leq [u]_{\mathcal{S}}$. Since $u - u_{\mathcal{R}} \leq v$ this implies the inequality on the right hand side of (4.6). The inequality on the left hand side is obvious. \square

We finish this section with the following lemma which is used in the proof of the main result.

Lemma 4.3. *Let $u \in \mathcal{U}_q$ and let A, B be two disjoint $C_{2/q, q'}$ -finely closed subsets of $\partial\Omega$. If $u \text{ supp}_{\partial\Omega}^q u \subset A \cup B$ and $[u]_A, [u]_B$ are σ -moderate then u is σ -moderate. Furthermore*

$$(4.7) \quad u = [u]_A \oplus [u]_B = [\max(u_A, u_B)]_{\#}.$$

Proof. Let τ and τ' be q -good positive measures such that $q\text{-supp } \tau \cap q\text{-supp } \tau' = \emptyset$. Then

$$[\max(u_{\tau}, u_{\tau'})]_{\#} = u_{\tau} \oplus u_{\tau'} = u_{\tau + \tau'}.$$

Let $\{\tau_n\}$ and $\{\tau'_n\}$ be increasing sequences of q -good measures such that

$$u_{\tau_n} \uparrow [u]_A, \quad u_{\tau'_n} \uparrow [u]_B.$$

By [35, Theorem 4.4]

$$(4.8) \quad \max([u]_A, [u]_B) \leq u \leq [u]_A + [u]_B.$$

Therefore

$$\max(u_{\tau_n}, u_{\tau'_n}) \leq u \implies u_{\tau_n + \tau'_n} \leq u.$$

On the other hand

$$u - u_{\tau_n + \tau'_n} \leq ([u]_A - u_{\tau_n}) + ([u]_B - u_{\tau'_n}) \downarrow 0.$$

Thus

$$(4.9) \quad \lim u_{\tau_n + \tau'_n} = u$$

so that u is σ -moderate.

Assertion (4.7) is equivalent to the statements: (a) u is the largest solution dominated by $[u]_A + [u]_B$ and (b) u is the smallest solution dominating $\max(u_A, u_B)$. Since the maximal solution U_F of a $C_{2/q, q'}$ -finely closed set $F \subset \partial\Omega$ is σ -moderate:

$$[u]_F = \sup\{v \in \mathcal{U}_q : v \leq u, v \text{ moderate, } \text{supp}_{\partial\Omega}^q v \subset F\}.$$

Suppose that $w \in \mathcal{U}_q$ and

$$u \leq w \leq [u]_A + [u]_B.$$

Then,

$$[w]_A \leq [u]_A, [w]_B \leq [u]_B \implies v \leq [u]_A.$$

Therefore, as $u \leq w$ we obtain,

$$[w]_A = [u]_A, \quad [w]_B = [u]_B.$$

Since u is σ -moderate, these equalities and (4.9) imply that $u = w$. This proves (a); statement (b) is proved in a similar way. \square

5. CHARACTERIZATION OF POSITIVE SOLUTIONS OF $-\Delta u + u^q = 0$.

In this section we present the main result of the paper:

Theorem 5.1. *Every positive solution of (1.1) is σ -moderate.*

The proof is based on several lemmas.

The following notation is used throughout the section: u is a positive solution of (1.1),

$$V := u^{q-1}, \quad L^V = -\Delta v + Vv = 0.$$

Thus V satisfies (2.1) and $L^V u = 0$. Therefore there exists a positive measure $\mu \in \mathfrak{M}$ such that

$$u = \mathbb{K}_\mu^V.$$

For any Borel set $E \subset \partial\Omega$ put

$$\mu_E = \mu \chi_E \text{ and } (u)_E = \mathbb{K}_{\mu_E}^V.$$

Lemma 5.2. *Let D be a C^2 domain such that $D \Subset \Omega$ and let $h \in L^1(\partial D)$, $0 \leq h \leq u$. Then*

$$(5.1) \quad S^V(D, h) \leq S_q(D, h).$$

Proof. Put $w := S_q(D, h)$ and $v := S^V(D, h)$. Then $w \leq u$ and consequently (recall that $V = u^{q-1}$)

$$0 = -\Delta w + w^q \leq -\Delta w + Vw.$$

Thus w is an L^V superharmonic in D such that $u = h$ on ∂D . On the other hand v is an L^V harmonic in D satisfying the same boundary condition. This implies (5.1). \square

Lemma 5.3. *If F is a compact subset of $\partial\Omega$ then*

$$(5.2) \quad (u)_F \leq [u]_F.$$

Proof. Let A be a Borel subset of $\partial\Omega$. Put

$$A_\beta = \{x \in \Omega : \rho(x) = \beta, \sigma(x) \in A\}$$

and

$$v_\beta^A = S^V(D_\beta, u \chi_{A_\beta}), \quad w_\beta^A = S_q(D_\beta, u \chi_{A_\beta}).$$

By Lemma 5.2, $v_\beta^A \leq w_\beta^A \leq u$. For any sequence $\{\beta_n\}$ decreasing to zero one can extract a subsequence $\{\beta_{n'}\}$ such that $\{w_{\beta_{n'}}^A\}$ and $\{v_{\beta_{n'}}^A\}$ converge locally uniformly; we denote the limits by w^A and v^A respectively. (The limits may depend on the sequence.) Then w^A is a solution of (1.1) while v^A is an L^V harmonic, and

$$(5.3) \quad v^A \leq w^A \leq [u]_{\tilde{Q}} \quad \forall Q \text{ open}, A \subset Q.$$

The second inequality follows from the fact that $w^A \leq u$ and w^A vanishes on $\partial\Omega \setminus \tilde{Q}$.

We apply the same procedure to the set $B = \partial\Omega \setminus A$ extracting a further subsequence of $\{\beta_{n'}\}$ in order to obtain the limits v^B and w^B . Thus

$$v^B \leq w^B \leq [u]_{\tilde{Q}'} \quad \forall Q' \text{ open}, B \subset Q'.$$

Note that

$$v^A + v^B = u, \quad v^A \leq \mathbb{K}_{\mu_{\tilde{Q}}}^V, \quad v^B \leq \mathbb{K}_{\mu_{\tilde{Q}'}}^V.$$

Therefore

$$(5.4) \quad v^A = u - v^B \geq \mathbb{K}_{\mu_{\partial\Omega \setminus \tilde{Q}'}}^V.$$

Now, given F compact, let A be a closed set and O an open set such that $F \subset O \subset A$ and let $B = \partial\Omega \setminus A$. Note that $\bar{B} \cap F = \emptyset$. By (5.4) with $Q' = B$

$$v^A \geq \mathbb{K}_{\mu_O}^V.$$

By (5.3)

$$v^A \leq w^A \leq [u]_{\tilde{Q}} \quad \forall Q \text{ open}, A \subset Q$$

and consequently

$$(5.5) \quad (u)_F \leq \mathbb{K}_{\mu_{\chi_O}}^V \leq [u]_{\tilde{Q}}.$$

If Q shrinks to F then $[u]_{\tilde{Q}} \downarrow [u]_F$ (see [35, Theorem 4.4]). Therefore (5.5) implies (5.2). \square

Lemma 5.4. *If $E \subset \partial\Omega$ is a Borel set and $C_{2/q,q'}(E) = 0$ then $\mu(E) = 0$.*

Proof. If F is a compact subset of E , $C_{2/q,q'}(F) = 0$ and therefore the removability theorem [33] implies that $[u]_F = 0$. Therefore, by Lemma 5.3, $(u)_F = 0$. Consequently $\mu(F) = 0$. As this holds for every compact subset of E we conclude that $\mu(E) = 0$. \square

Lemma 5.5. *Let $\nu \in W^{-2/q,q}(\partial\Omega)$ be a positive measure and let u_ν be the solution of (1.1) with trace ν . Suppose that there exists no positive solution of (1.1) dominated by the supersolution $v = \inf(u, \mathbb{K}_\nu)$. Then $\mu \perp \nu$.*

Proof. First we show:

Assertion 1. If $V' := v^{q-1}$ then v is an $L^{V'}$ superharmonic and furthermore it is an $L^{V'}$ potential.

Since v is a supersolution of (1.1)

$$0 \leq -\Delta v + v^q = -\Delta v + V'v.$$

Thus v is an $L^{V'}$ superharmonic. Suppose that there exists a positive $L^{V'}$ harmonic w such that $w \leq v$. Then w is a subsolution of (1.1):

$$-\Delta w + w^q \leq -\Delta w + V'w = 0.$$

This implies that there exists a positive solution of (1.1) dominated by v , contrary to assumption. Thus v is an $L^{V'}$ -potential.

Note that

$$\int_{\partial\Omega} \mathbb{K}_\nu V' \rho \, dx \leq \int_{\partial\Omega} (\mathbb{K}_\nu)^q \rho \, dx < \infty.$$

Therefore \mathbb{K}_ν is an $L^{V'}$ superharmonic satisfying (3.4). By Lemma 3.4 (i), the largest $L^{V'}$ harmonic dominated by \mathbb{K}_ν , say w , is $L^{V'}$ moderate and has m-boundary trace ν . This implies that

$$\mathbb{K}_\nu - w =: p$$

is an $L^{V'}$ -potential. w can be represented in the form

$$w = \mathbb{K}_{\nu'}^{V'}$$

where ν' is a positive finite measure on $\partial\Omega$ and, by Proposition 3.8, ν , ν' are mutually a.c.

By the relative Fatou theorem, since v, p are $L^{V'}$ potentials and w is an $L^{V'}$ harmonic,

$$(5.6) \quad v/w \rightarrow 0, \quad \mathbb{K}_\nu/w \rightarrow 1 \quad L^{V'} - \text{finely } \nu' \text{-a.e.}$$

Since $v = \inf(u, \mathbb{K}_\nu)$, (5.6) implies that

$$(5.7) \quad u/w \rightarrow 0 \quad L^{V'} - \text{finely } \nu' \text{-a.e.}$$

Further, by (5.6) and (5.7)

$$(5.8) \quad u/\mathbb{K}_\nu \rightarrow 0 \quad L^{V'} - \text{finely } \nu' \text{-a.e.}$$

Since ν, ν' are mutually a.c., ' ν -a.e.' is equivalent to ' ν' -a.e.'. Therefore, in view of Proposition 2.8, (5.8) implies

$$(5.9) \quad u/\mathbb{K}_\nu \rightarrow 0 \quad \text{n.t. } \nu \text{-a.e.}$$

However, \mathbb{K}_ν is also an L^V superharmonic. Therefore \mathbb{K}_ν can be represented in the form

$$\mathbb{K}_\nu = w^* + p^*,$$

where w^* is an L^V -harmonic and p^* an L^V -potential. Let $\tau \in \mathfrak{M}$ be the L^V trace of w^* , i.e., $w^* = \mathbb{K}_\tau^V$. Then, by the relative Fatou theorem,

$$\mathbb{K}_\nu/u \rightarrow \frac{d\tau}{d\mu} =: h \quad L^V - \text{finely, } \mu \text{-a.e.}$$

and therefore, by Proposition 2.8,

$$(5.10) \quad \mathbb{K}_\nu/u \rightarrow h \quad \text{n.t. } \mu \text{-a.e.}$$

Since $0 \leq h < \infty$ μ -a.e., (5.9) and (5.10) imply that $\nu \perp \mu$. \square

Lemma 5.6. *Suppose that for every positive measure $\nu \in W^{-2/q,q}(\partial\Omega)$, there exists no positive solution of (1.1) dominated by $v = \inf(u, \mathbb{K}_\nu)$. Then $u = 0$.*

Proof. By Lemma 5.5,

$$\mu \perp \nu \quad \forall \nu \in W^{-2/q,q}(\partial\Omega), \quad \nu \geq 0.$$

Suppose that $\mu \neq 0$. By Lemma 5.4, μ vanishes on sets of $C_{2/q,q'}$ zero. Therefore (by Feyel and de la Pradelle [19] or Dal Maso [10]) μ is the limit of an increasing sequence $(\mu_k) \subset W_+^{-2/q,q}$. For every k there exists a Borel set $A_k \subset \partial\Omega$ such that,

$$\mu(A_k) = 0, \quad \mu_k(\partial\Omega \setminus A_k) = 0.$$

Therefore, if $A = \cup A_k$ and $A' = \partial\Omega \setminus A$ then

$$\mu(A) = 0, \quad \mu_k(A') = 0 \quad \forall k.$$

Since $\mu_k \leq \mu$ we have $\mu_k(A) = 0$ and therefore $\mu_k = 0$. Contradiction! \square

PROOF OF THEOREM 5.1 Let $\{Q_n\}$ be a regular decomposition of $\mathcal{R}(u)$ and put

$$v_n := [u]_{\tilde{Q}_n}.$$

Using the notation introduced in (4.4), v_n is moderate with boundary trace ν_n and

$$v_n \uparrow u_{\mathcal{R}}.$$

Thus the solution $u_{\mathcal{R}}$ is σ -moderate and

$$u \ominus u_{\mathcal{R}} \leq [u]_{\mathcal{S}(u)} =: u_{\mathcal{S}}.$$

Assertion 1 $u_{\mathcal{S}}$ is σ -moderate.

Before proving the assertion we verify that it implies that u is σ -moderate. Put

$$u_n := v_n \oplus u_{\mathcal{S}}.$$

By Lemma 4.3, as $\tilde{Q}_n \cap \mathcal{S}(u) = \emptyset$, it follows that u_n is σ -moderate. As $\{u_n\}$ is increasing it follows that $\bar{u} = \lim u_n$ is a σ -moderate solution of (1.1). In addition

$$[\max(v_n, u_{\mathcal{S}})]_{\#} = u_n = v_n \oplus u_{\mathcal{S}} \implies \max(u_{\mathcal{R}}, u_{\mathcal{S}}) \leq \bar{u} \leq u_{\mathcal{R}} + u_{\mathcal{S}}.$$

This further implies that $\mathcal{S}(u) = \mathcal{S}(\bar{u})$ and that $\text{tr } \bar{u} = \text{tr } u$. By uniqueness of the σ -moderate solution we conclude that $u = \bar{u}$.

We turn to the proof of *Assertion 1*. To simplify notation, we put $u = u_{\mathcal{S}}$ and denote $F := \text{supp}_{\partial\Omega}^q u$. (Incidentally, $F \subset \mathcal{S}(u)$ but it is possible that there is no equality. In fact F consists precisely of the $C_{2/q,q'}$ -thick points of $\mathcal{S}(u)$. The set $\mathcal{S}(u) \setminus F$ is contained in the singular set of $u_{\mathcal{R}}$.)

For $\nu \in W^{-2/q,q}$ we denote by u_ν the solution of (1.1) with boundary trace ν . Put

$$(5.11) \quad u^* = \sup\{u_\nu : \nu \in W^{-2/q,q}, 0 < u_\nu \leq u\}.$$

By Lemma 5.6 the family over which the supremum is taken is not empty. Therefore u^* is a positive solution of (1.1) and it is well-known that it is σ -moderate. By its definition, $u^* \leq u$.

Let $F^* = \text{supp}_{\partial\Omega}^q u^*$. Then F^* is $C_{2/q,q'}$ -finely closed and $F^* \subset F$. Suppose that $C_{2/q,q'}(F \setminus F^*) > 0$. Then there exists a compact set $E \subset F \setminus F^*$ such that $C_{2/q,q'}(E) > 0$ and $\partial\Omega \setminus F^* =: Q^*$ is a $C_{2/q,q'}$ -finely open set containing E . Furthermore there exists a $C_{2/q,q'}$ -finely open set Q' such that $E \subset Q' \subset \tilde{Q}' \subset Q^*$ ([35, Lemma 2.4]). Since $Q' \subset \text{supp}_{\partial\Omega}^q u$, $[u]_{\tilde{Q}'} > 0$ and therefore, by Lemma 5.6, there exists a positive measure $\tau \in W^{-2/q,q}$ supported in \tilde{Q}' such that $u_\tau \leq u$. As the q -supp τ is a $C_{2/q,q'}$ -finely closed set disjoint from F^* it follows that $u^* \not\leq u_\tau$. On the other hand, since $\tau \in W^{-2/q,q}$ and $u_\tau \leq u$, it follows that $u_\tau \leq u^*$. This contradiction shows that

$$(5.12) \quad C_{2/q,q'}(F \setminus F^*) = 0.$$

Further u^* is σ -moderate and therefore there exists a $C_{2/q,q'}$ -finely closed set $F_0^* \subset F^*$ such that $\mathcal{S}(u^*) = F_0^*$ and $\mathcal{R}(u^*) = \partial\Omega \setminus F_0^*$. Suppose that $C_{2/q,q'}(F \setminus F_0^*) > 0$ and put $Q_0 := \partial\Omega \setminus F_0^*$. Let $E \subset F \setminus F_0^*$ be a compact set such that $C_{2/q,q'}(E) > 0$ and let Q' be a $C_{2/q,q'}$ -finely open set such that $E \subset Q' \subset \tilde{Q}' \subset Q_0$. Then $\tilde{Q}' \subset \mathcal{R}(u^*)$ and consequently $[u^*]_{\tilde{Q}'}$ is a moderate solution of (1.1), i.e.

$$[u^*]_{\tilde{Q}'} \in L_\rho^q(\Omega).$$

On the other hand Q' is a $C_{2/q,q'}$ -finely open neighborhood of E which is a non-empty subset of $F = \text{supp}_{\partial\Omega}^q u$; therefore $[u]_{\tilde{Q}'}$ is a purely singular solution of (1.1), i.e.,

$$\int_\Omega ([u]_{\tilde{Q}'})^q \rho \, dx = \infty, \quad \mathcal{S}([u]_{\tilde{Q}'}) = \text{supp}_{\partial\Omega}^q [u]_{\tilde{Q}'}.$$

It follows that $v := [[u]_{\tilde{Q}'} - [u^*]_{\tilde{Q}'}]_\#$ is a purely singular solution of (1.1).

Let v^* be defined as in (5.11) with u replace by v . Then v^* is a singular, σ -moderate solution of (1.1). Since $v^* \leq u$ and it is σ -moderate it follows that $v^* \leq u^*$. On the other hand, since v^* is singular and $\text{supp}_{\partial\Omega}^q v^* \subset \tilde{Q}' \subset \mathcal{R}(u^*)$ it follows that $u^* \not\leq v^*$, i.e. $(v^* - u^*)_+$ is not identically zero. Since both u^* and v^* are σ -moderate, it follows that there exists $\tau \in W^{-2/q,q}$ such that $u_\tau \leq v^*$ but $(u_\tau - u^*)_+$ is not identically zero. Therefore $u^* \not\leq \max(u^*, u_\tau)$. The function $\max(u^*, u_\tau)$ is a subsolution of (1.1) and the smallest solution above it, which we denote by Z is *strictly larger* than u^* . However $u_\tau \leq v^* \leq u^*$ and consequently $Z = u^*$. This contradiction proves that

$$(5.13) \quad C_{2/q,q'}(F \setminus F_0^*) = 0$$

In conclusion, u^* is σ -moderate, $\text{supp}_{\partial\Omega}^q u^* \subset F$ and $F_0^* = \mathcal{S}(u^*)$ is $C_{2/q,q'}$ -equivalent to F . Therefore, by Theorem 4.1, $u^* = U_F$, the maximal solution supported in F . Since, by definition $u^* \leq u$, it follows $u^* = u$. \square

REFERENCES

- [1] Adams D. R. and Hedberg L. I., Function spaces and potential theory, Grundlehren Math. Wissen. **314**, Springer (1996).
- [2] Ancona A., *Theorie du potentiel sur les graphes et les varietes*, in Springer Lecture Notes No. 1427 (ed. P.L. Hennequin) p.1-112 (1988).
- [3] Ancona A., *Negatively curved manifolds, elliptic operators and the Martin boundary*, Annals of Mathematics, Second Series, **125**, 495-536 (1987).
- [4] Ancona A., Personal communication, 2005.
- [5] Ancona A., Personal communication, 2010.
- [6] Ancona A., *A necessary condition for the fine regularity of a boundary point with respect to a Schrödinger equation*, Appendix in ‘Boundary value problems with measures’, by L.Véron and C.Yarur (preprint).
- [7] Bandle C. and Marcus M. *Sur les solutions maximales de problmes elliptiques non linaires*, C. R. Acad. Sci. Paris Ser. I **311** (1990) 9193.
- [8] Bandle C. and Marcus M. *Asymptotic behaviour of solutions and their derivatives for semilinear elliptic problems with blow up on the boundary*, Ann. Inst. Poincar **12** (1995) 155171.
- [9] Baras P., Pierre M. *Singularités éliminables pour des équations semilinéaires*, Ann. Inst. Fourier **34** (1984), 185–206.
- [10] Dal Maso G. *On the integral representation of certain local functionals*, Ricerche Mat. **32 no. 1** (1983) 85-113.
- [11] Dynkin E. B., *A probabilistic approach to one class of nonlinear differential equations*, Probab. Th, Rel. Fields **89** (1991), 89–115.
- [12] Dynkin E. B., *Superprocesses and partial differential equations*, Ann. Probab **21** (1993), 1185–1262.
- [13] Dynkin E. B., *Stochastic boundary values and boundary singularities for solutions of the equation $Lu = u^\alpha$* , J. Funct. Anal. **153** (1998), 147–186.
- [14] Dynkin E. B. *Diffusions, Superdiffusions and Partial Differential Equations*, American Math. Soc., Providence, Rhode Island, Colloquium Publications **50**, 2002.
- [15] Dynkin E. B. *Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations*, American Math. Soc., Providence, Rhode Island, Colloquium Publications **34**, 2004.
- [16] Dynkin E. B. and Kuznetsov S. E. *Superdiffusions and removable singularities for quasilinear partial differential equations*, Comm. Pure Appl. Math. **49**, 125-176 (1996).
- [17] Dynkin E. B. and Kuznetsov S. E. *Trace on the boundary for solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. **350** (1998) 4499-4519.
- [18] Dynkin E. B. and Kuznetsov S. E. *Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations*, Comm. Pure Appl. Math. **51**, 897-936 (1998).
- [19] Feyel D. and de la Pradelle A. *Topologies fines et compactifications associes certains espaces de Dirichlet*, Ann. Inst. Fourier (Grenoble) **27**, 121146 (1977).
- [20] Gmira A. and Veron L., *Boundary singularities of solutions of nonlinear elliptic equations*, Duke J. Math. **64**, 271-324 (1991).
- [21] Kuznetsov S.E. *σ -moderate solutions of $Lu = u^\alpha$ and fine trace on the boundary*, C.R.Acad. Sc. Serie I **326**, 1189-1194 (1998).
- [22] Keller J. B. , *On solutions of $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10**, 503-510 (1957).

- [23] Le Gall J. F., *A class of path-valued Markov processes and its connections with partial differential equations.*, Probab. Th. Rel. Fields 102 (1993), 2546.
- [24] Le Gall J. F., *Solutions positives de $\Delta u = u^2$ dans le disque unité*, C.R. Acad. Sci. Paris, Sér. I 317 (1993), 873878.
- [25] Le Gall J. F., *The Brownian snake and solutions of $\Delta u = u^2$ in a domain*, Probab. Th. Rel. Fields **102**, 393-432 (1995).
- [26] Le Gall J. F., *A probabilistic Poisson representation for positive solutions of $\Delta u = u^2$ in a domain*, Comm. Pure Appl. Math. 50 (1997), 69103.
- [27] Le Gall J. F., *Spatial branching processes, random snakes and partial differential equations*, Birkhäuser, Basel/Boston/Berlin, 1999.
- [28] Loewner C. and Nirenberg L. *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, Orlando, FL (1974), 245272.
- [29] Marcus M. and Véron L., *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Annales de l'Institut Henri Poincaré 14 (1997), 237-274.
- [30] Marcus M. and Véron L., *Trace au bord des solutions positives d'équations elliptiques et paraboliques non linéaires. Résultats d'existence et d'unicité*, [Boundary trace of positive solutions of nonlinear parabolic and elliptic equations. Existence and uniqueness results.] C. R. Acad. Sci. Paris Sér. I, Math. **323**, no.6 (1996) 603608.
- [31] Marcus M. and Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. rat. Mech. Anal. **144**, 201-231 (1998).
- [32] Marcus M. and Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. **77**, 481-524 (1998).
- [33] Marcus M. and Véron L., *Removable singularities and boundary trace*, J. Math. Pures Appl. **80**, 879-900 (2000).
- [34] Marcus M. and Véron L., *Capacitary estimates of positive solutions of semilinear elliptic equations with absorption*, J. European Math. Soc. **6**, 483-527 (2004).
- [35] Marcus M. and Véron L., *The precise boundary trace of positive solutions of the equation $\Delta u = u^q$ in the supercritical case*, in "Perspectives in Nonlinear Partial Differential Equations" (in honor of H. Brezis), Berestycki et al eds., Contemporary Mathematics A. M. S. **446** (2007) p. 345-384.
- [36] Marcus M. and Veron L., *On a characterisation of Besov spaces with negative exponents*, Topics Around the Research of Vladimir Mazya /International Mathematical Series 11-13, Springer Science+Business Media, 2009.
- [37] Marcus M. and Veron L., *Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets*, J. Math. Pures Appl. **91** (2009) 256-295.
- [38] Mselati B., *Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation*, (English summary) Mem. Amer. Math. Soc. **168**, no. **798**, (2004) 121 pp.
- [39] Osserman R., *On the inequality $\Delta u \geq f(u)$* , Pacific J. Math. 7, 1641-1647 (1957).
- [40] Triebel H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Pub. Co., 1978.

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